2015.1.13
物性物理学C

非線形振動子と同期現象

Stuart-Landau方程式

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = ax - \omega y - \left(x^2 + y^2\right)\left(x - by\right) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = ay + \omega x - \left(x^2 + y^2\right)\left(y + bx\right) \end{cases}$$

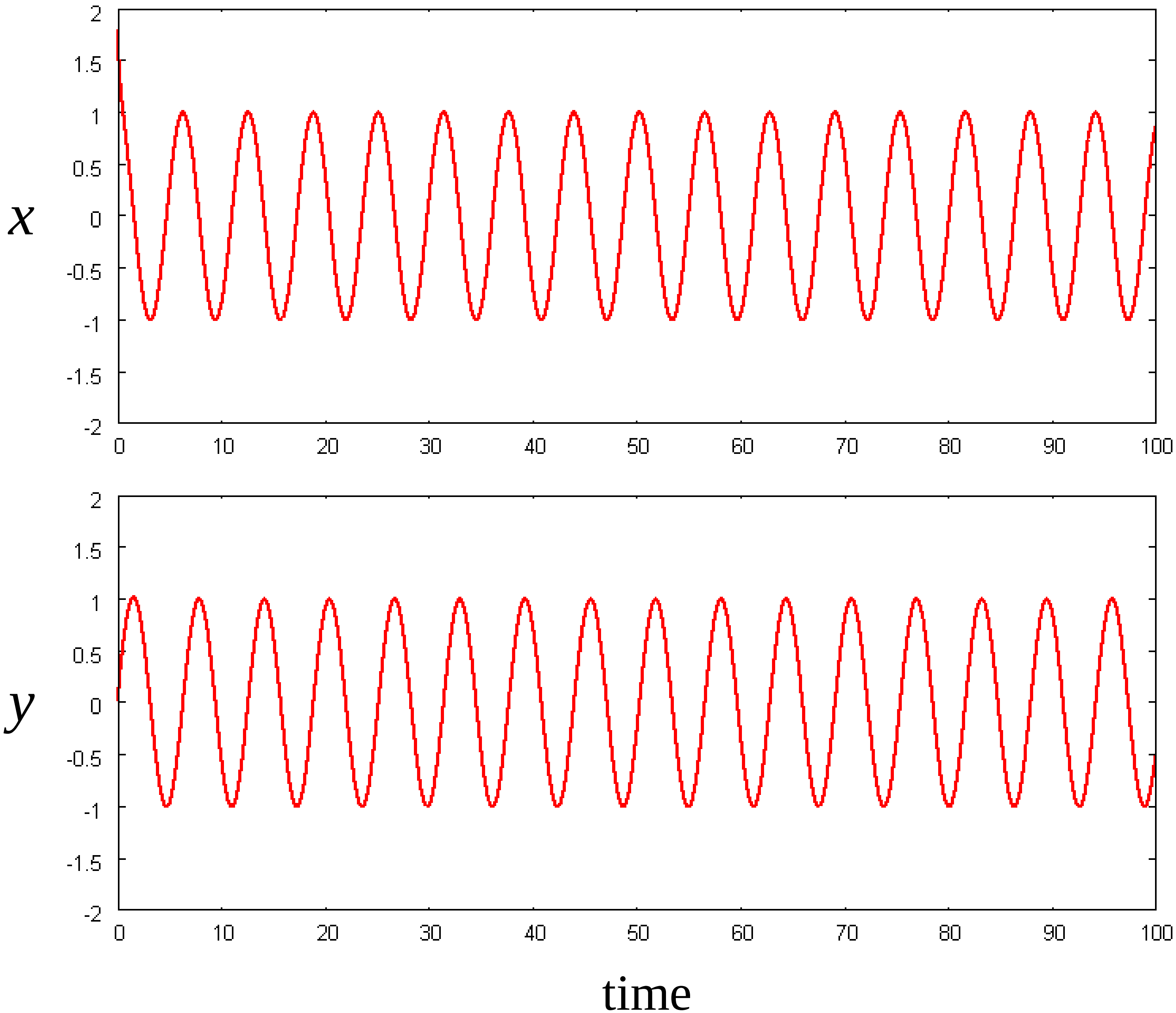
$$\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}t} = ar - r^3 \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \omega \end{cases}$$

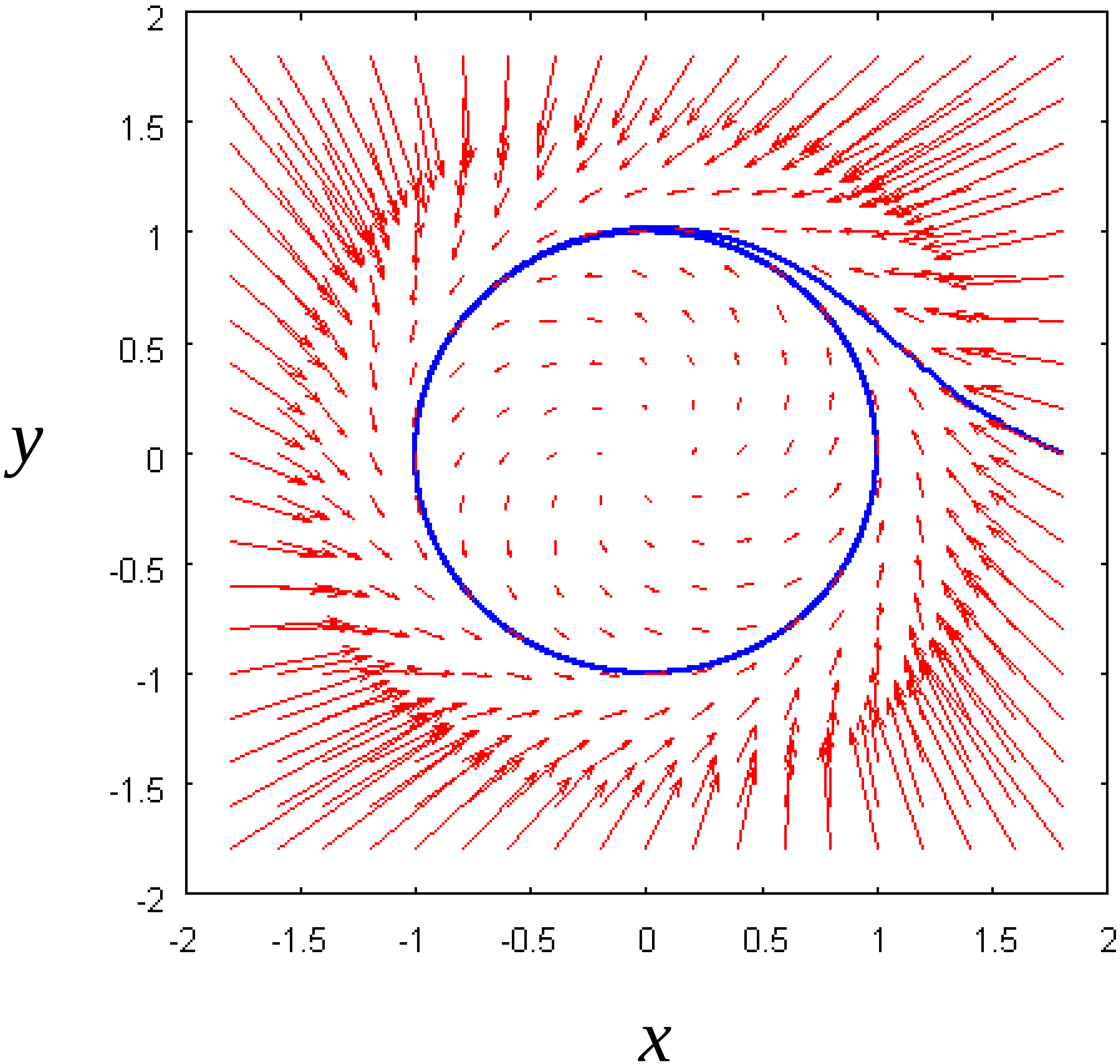
$$r^2 = x^2 + y^2$$

$$\frac{y}{x} = \tan\theta$$

$$a = 1$$

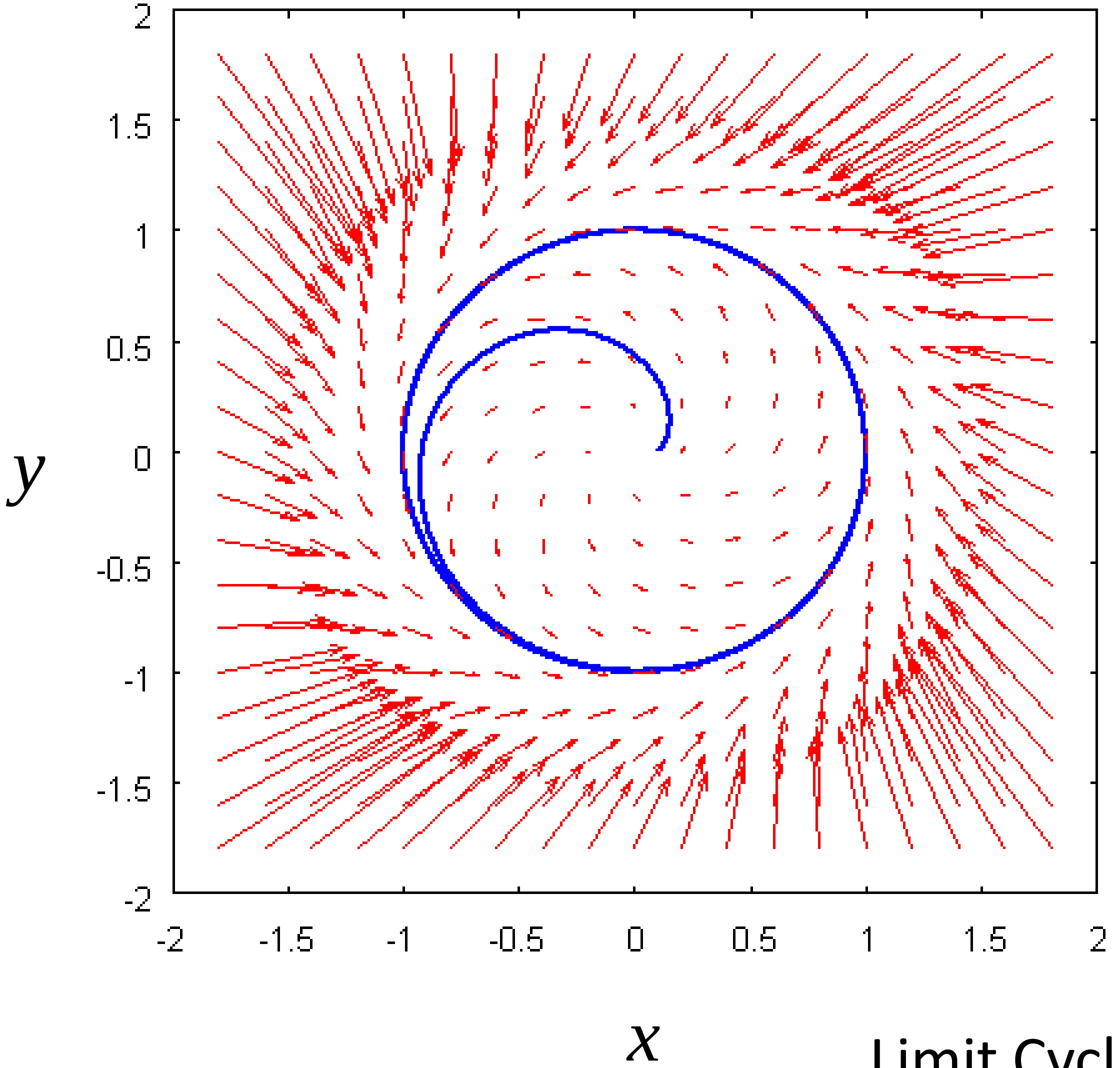
$$b = 0$$


x
y
time
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
10
20
30
40
50
60
70
80
90
100



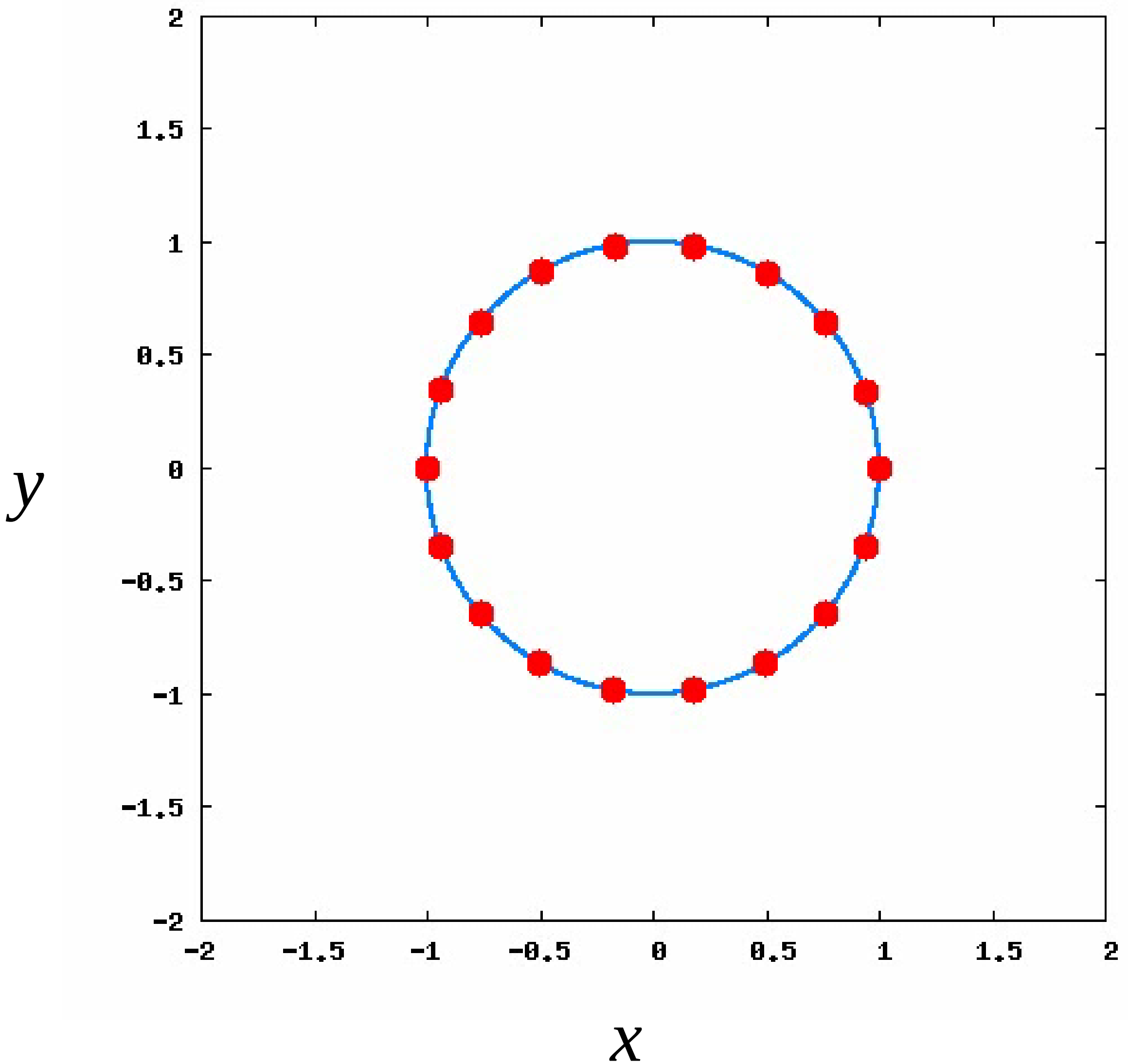
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
y
x

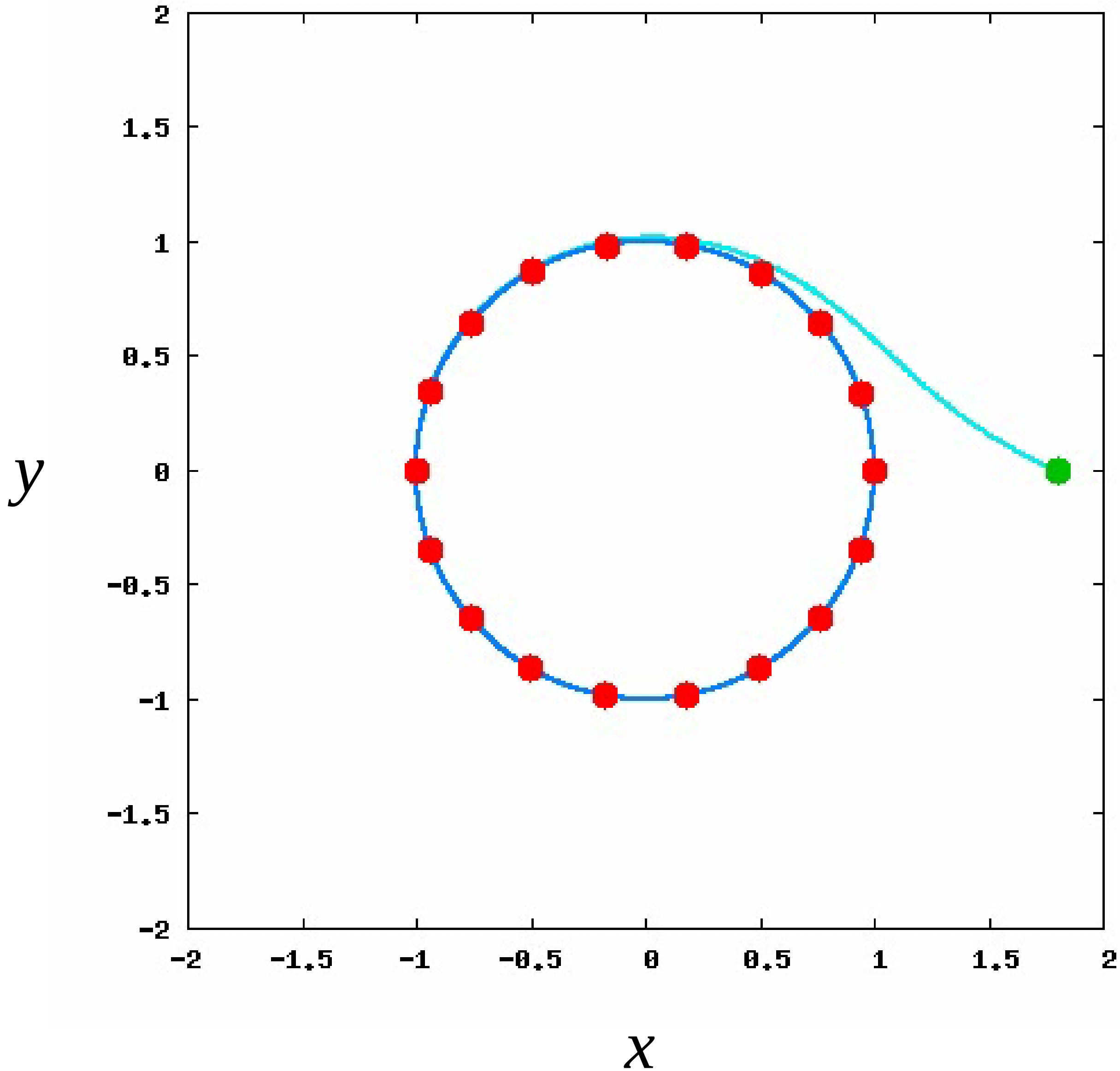
初期値を変えても

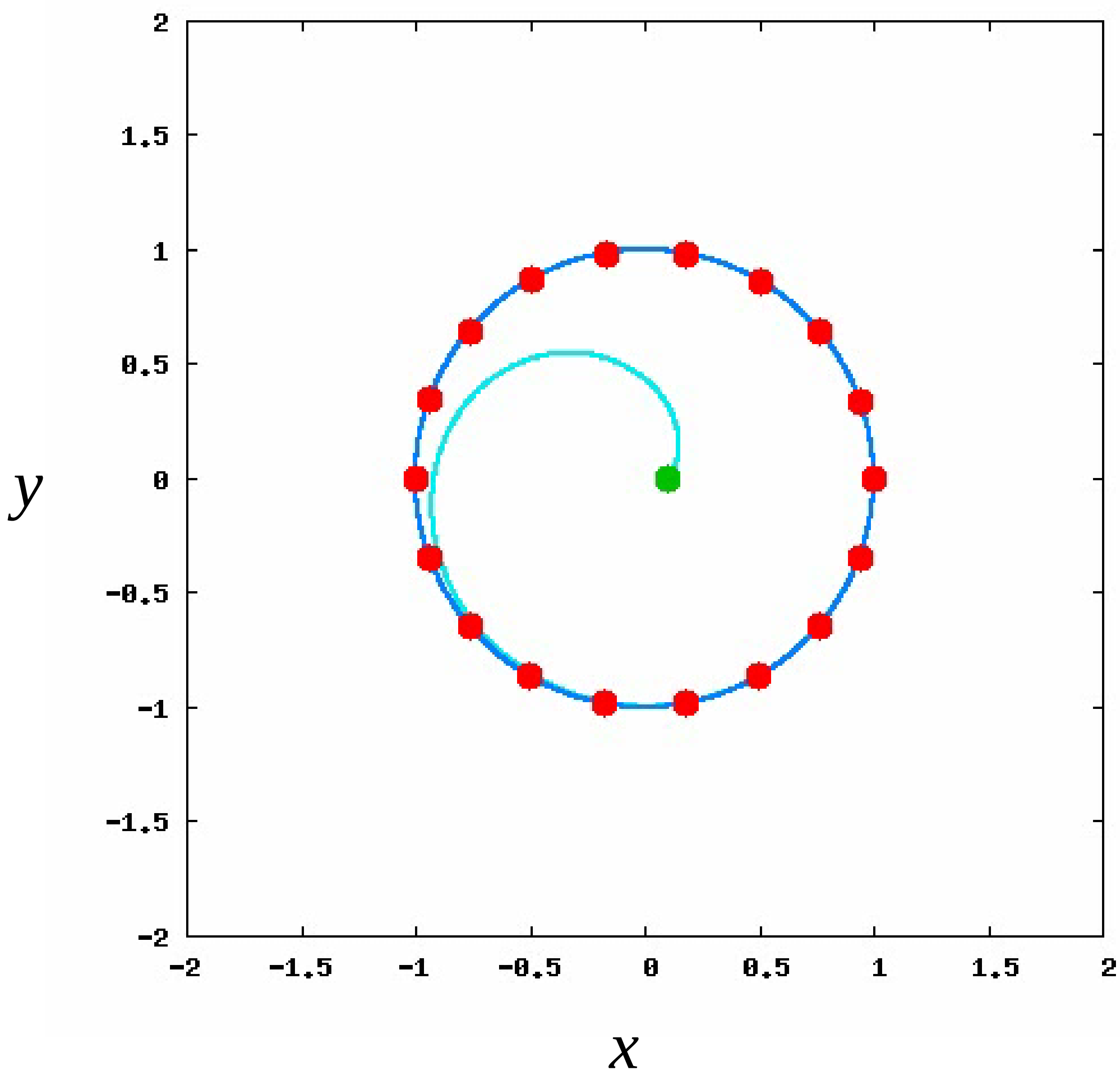


Limit Cycle (極限閉軌道)

リミットサイクル上の運動

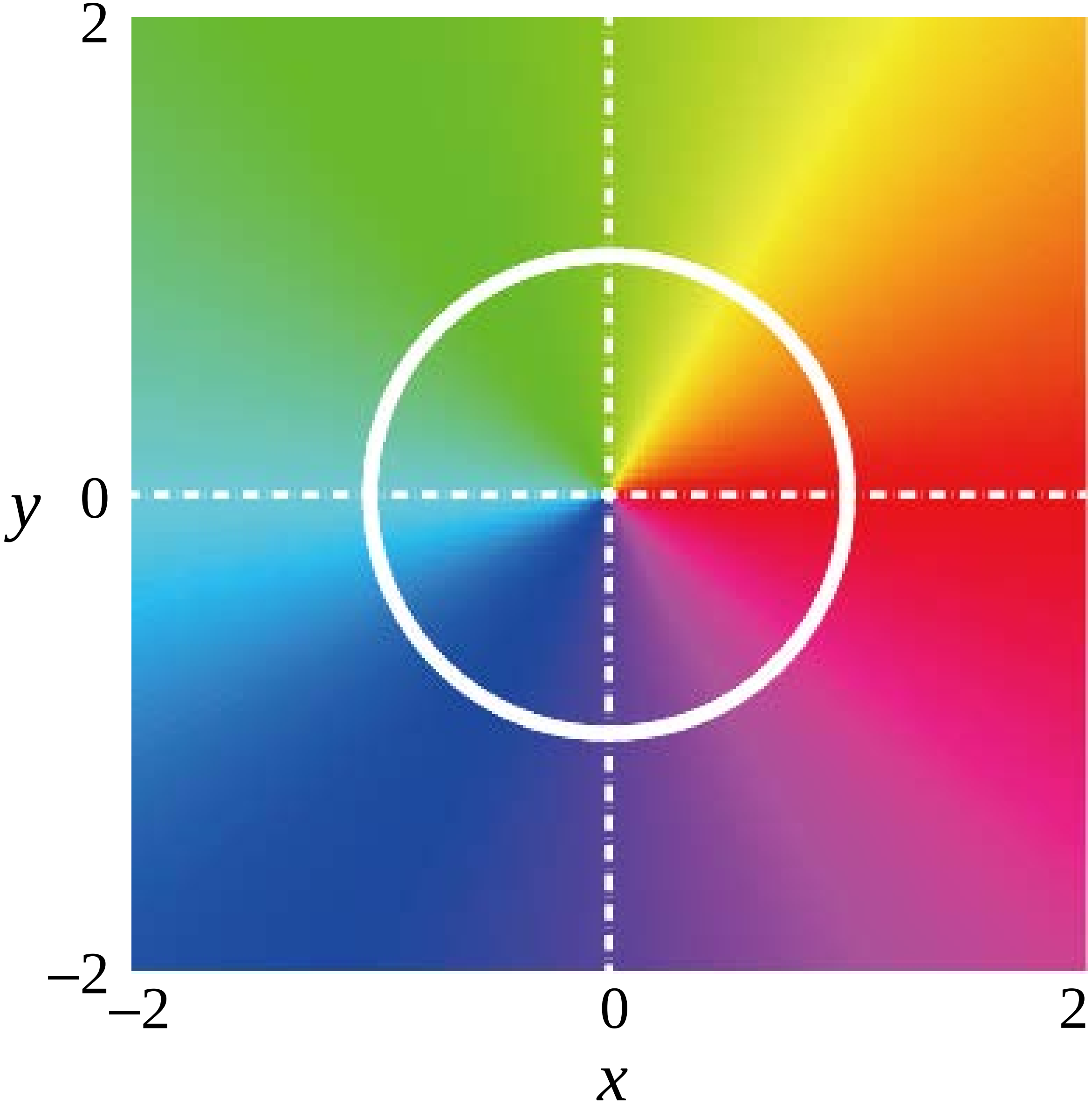



2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
y
x



2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
y
x

等位相面

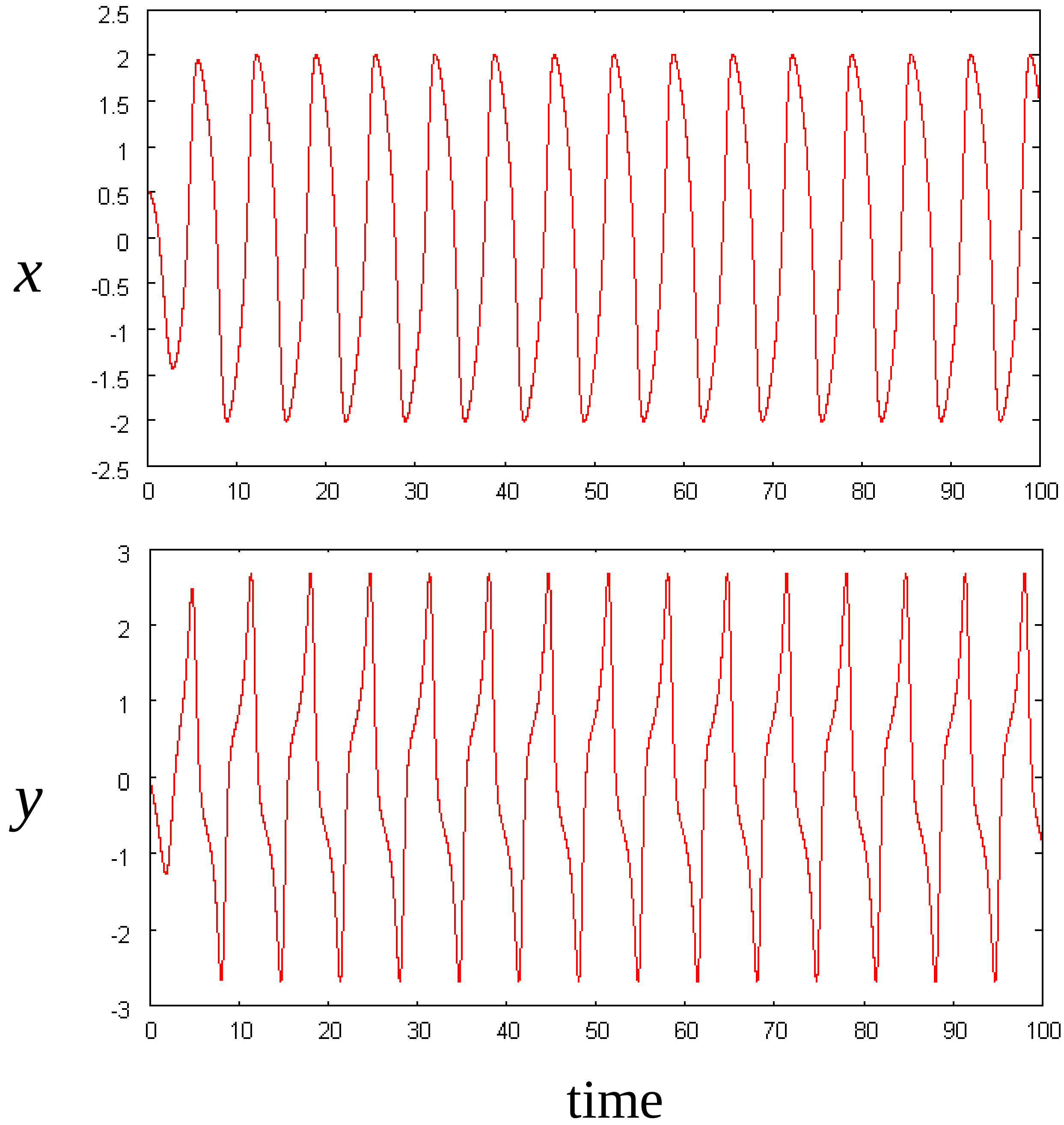


$$\theta = \arctan\left(\frac{y}{x}\right)$$

van der Pol 方程式　～　丸くなくても・・・

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \alpha\left(x^2 - 1\right)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0$$

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -\alpha\left(x^2 - 1\right)y - x \end{cases}$$


x
y
time
2.5
2
1.5
1
0.5
0
-0.5
-1
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-2
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10
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30
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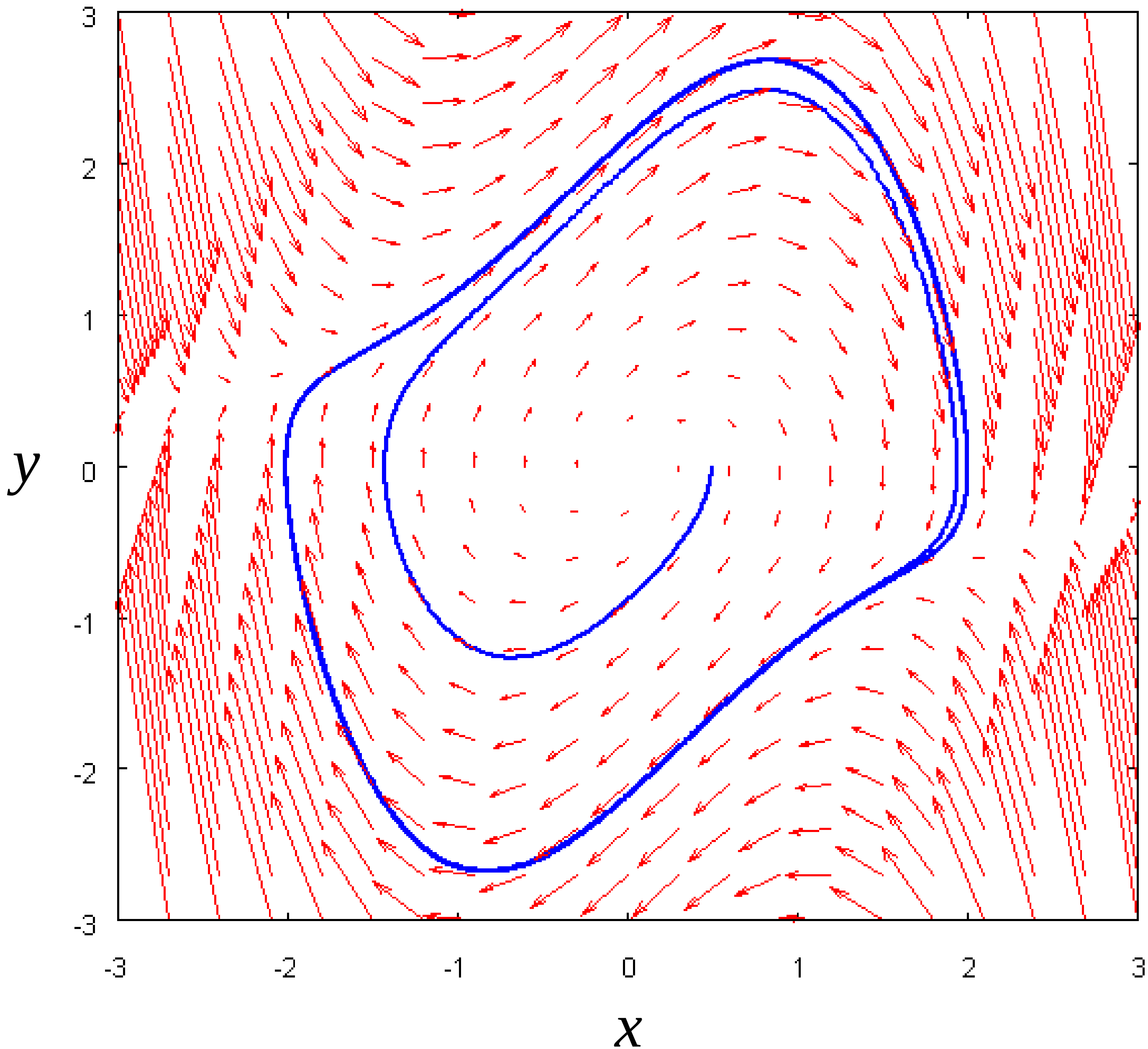
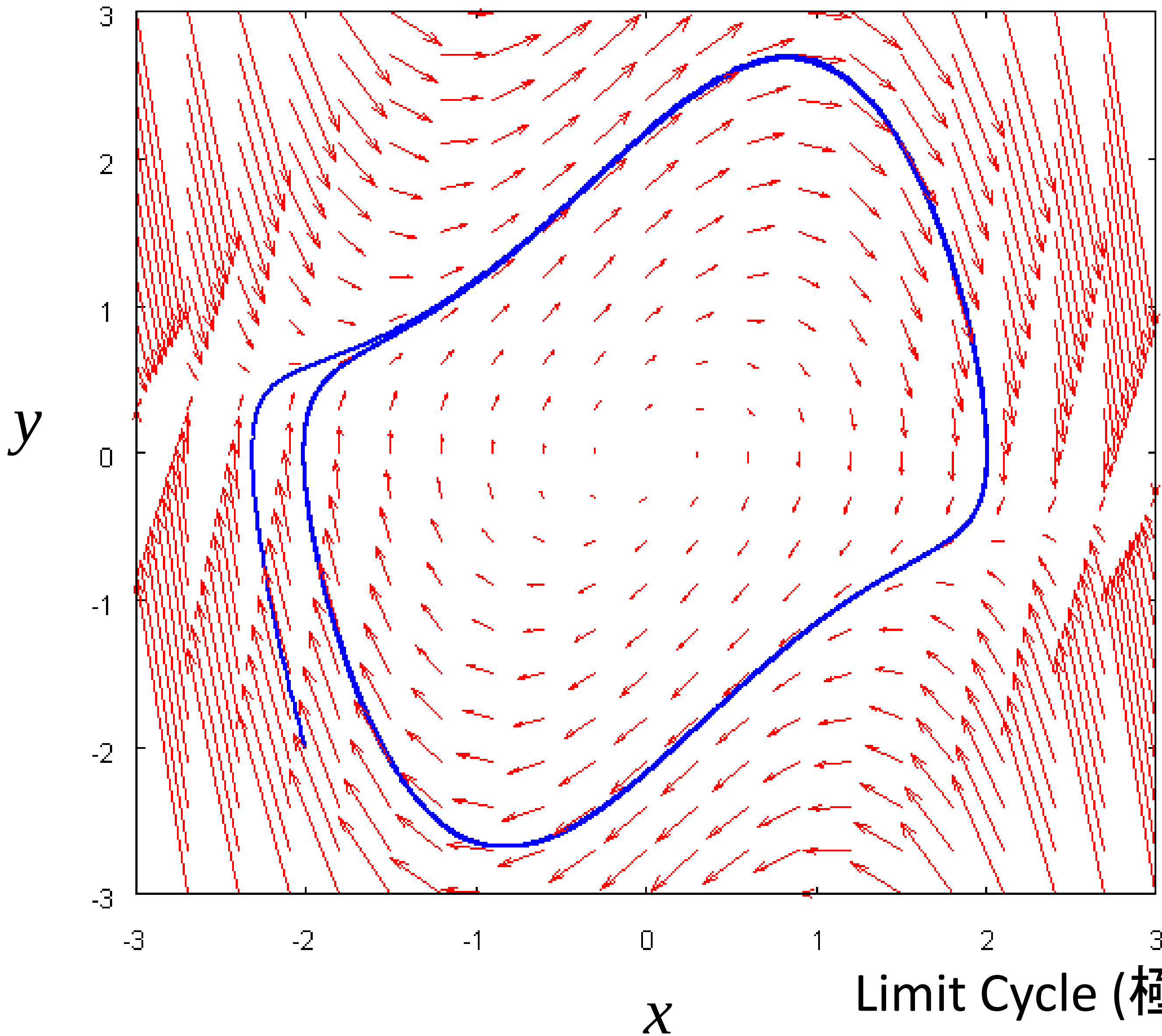
60
70
80
90
100

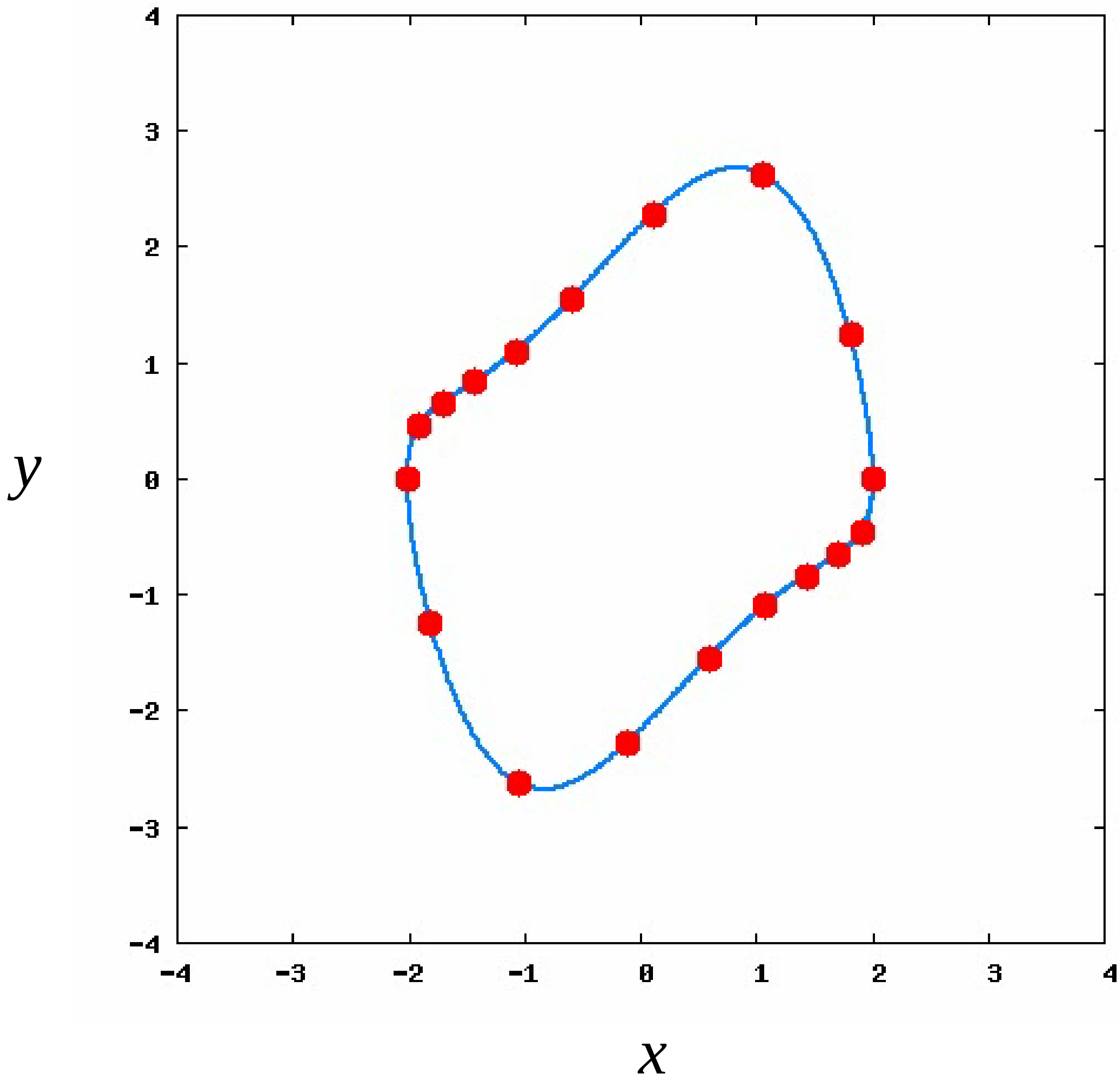


3
2
1
0
-1
-2
-3
y
-3
-2
-1
0
1
2
3
x

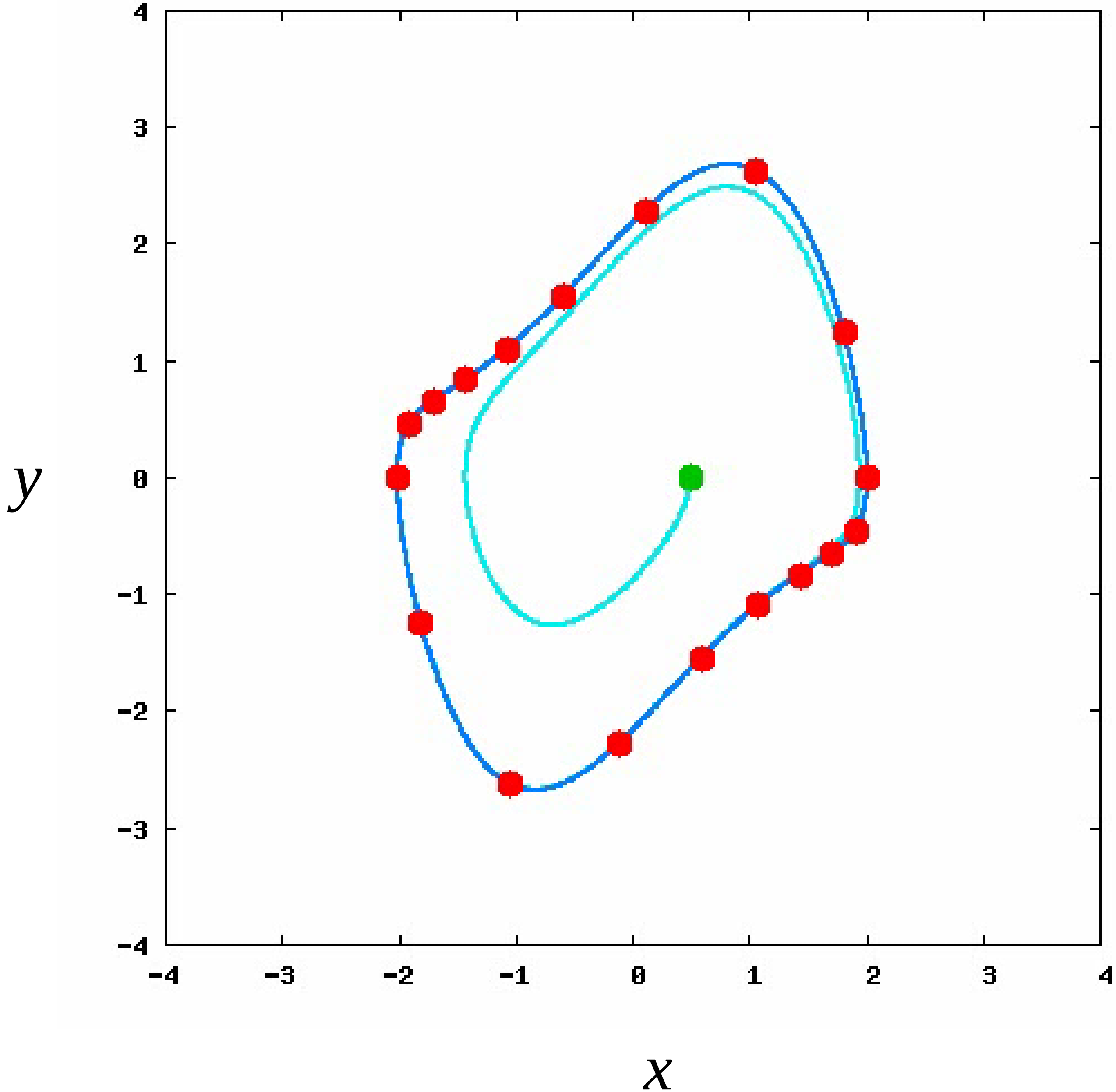
初期値を変えても



Limit Cycle (極限軌道)

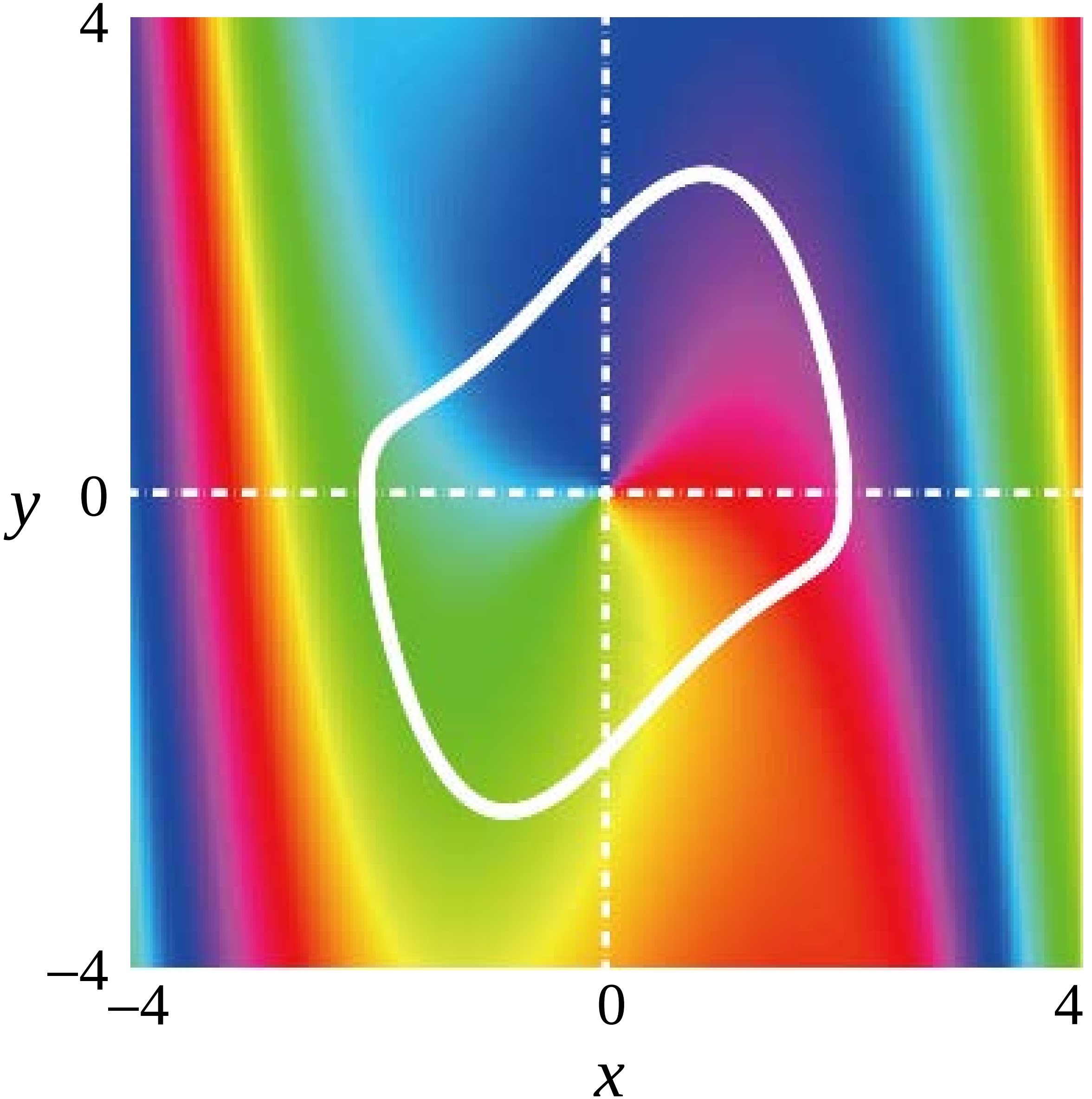


4
3
2
1
0
−1
−2
−3
−4
y
−4
−3
−2
−1
0
1
2
3
4
x



4
3
2
1
0
−1
−2
−3
−4
y
−4
−3
−2
−1
0
1
2
3
4
x

等位相面



さまざまな非線形振動子

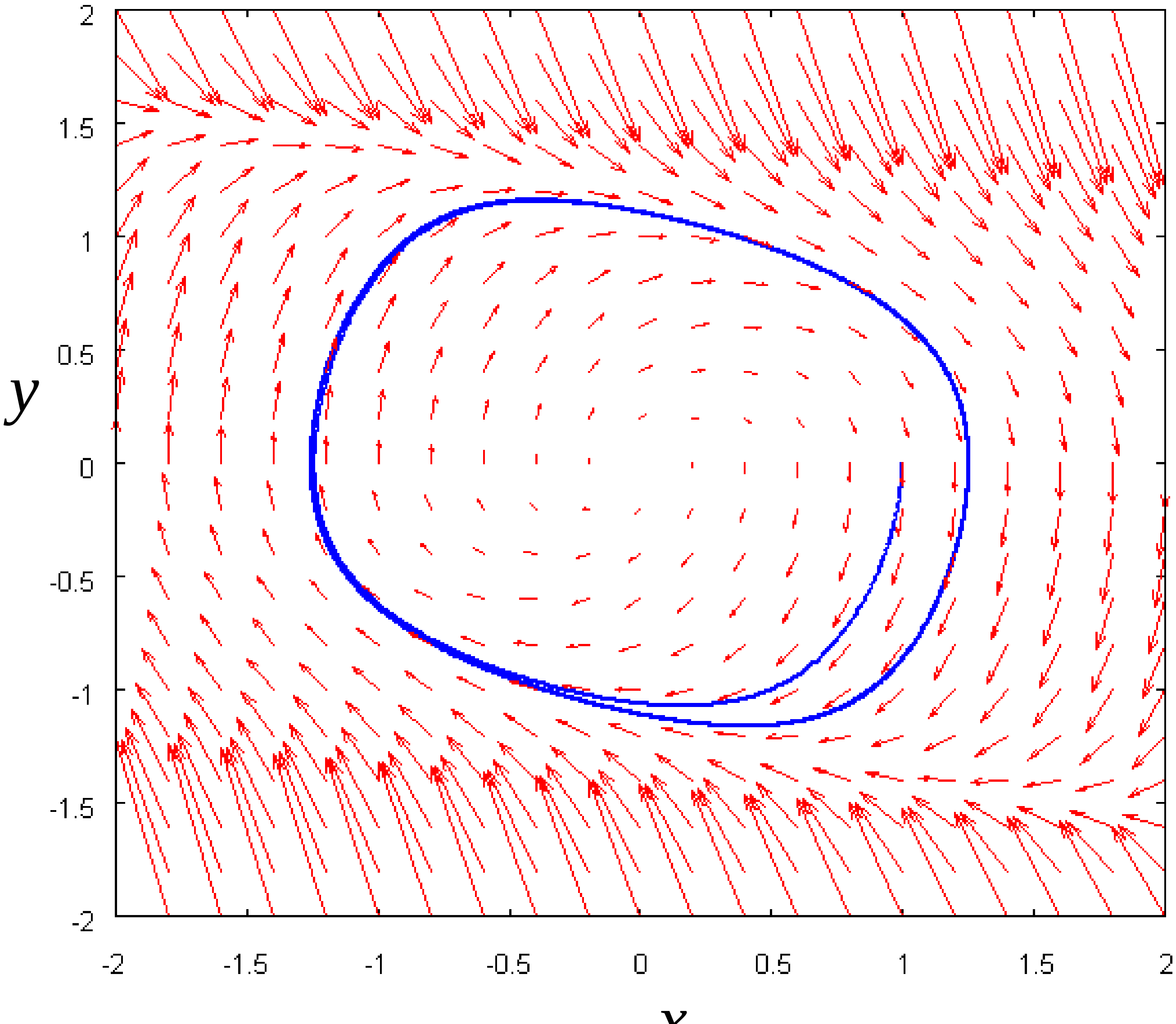
Rayleigh方程式

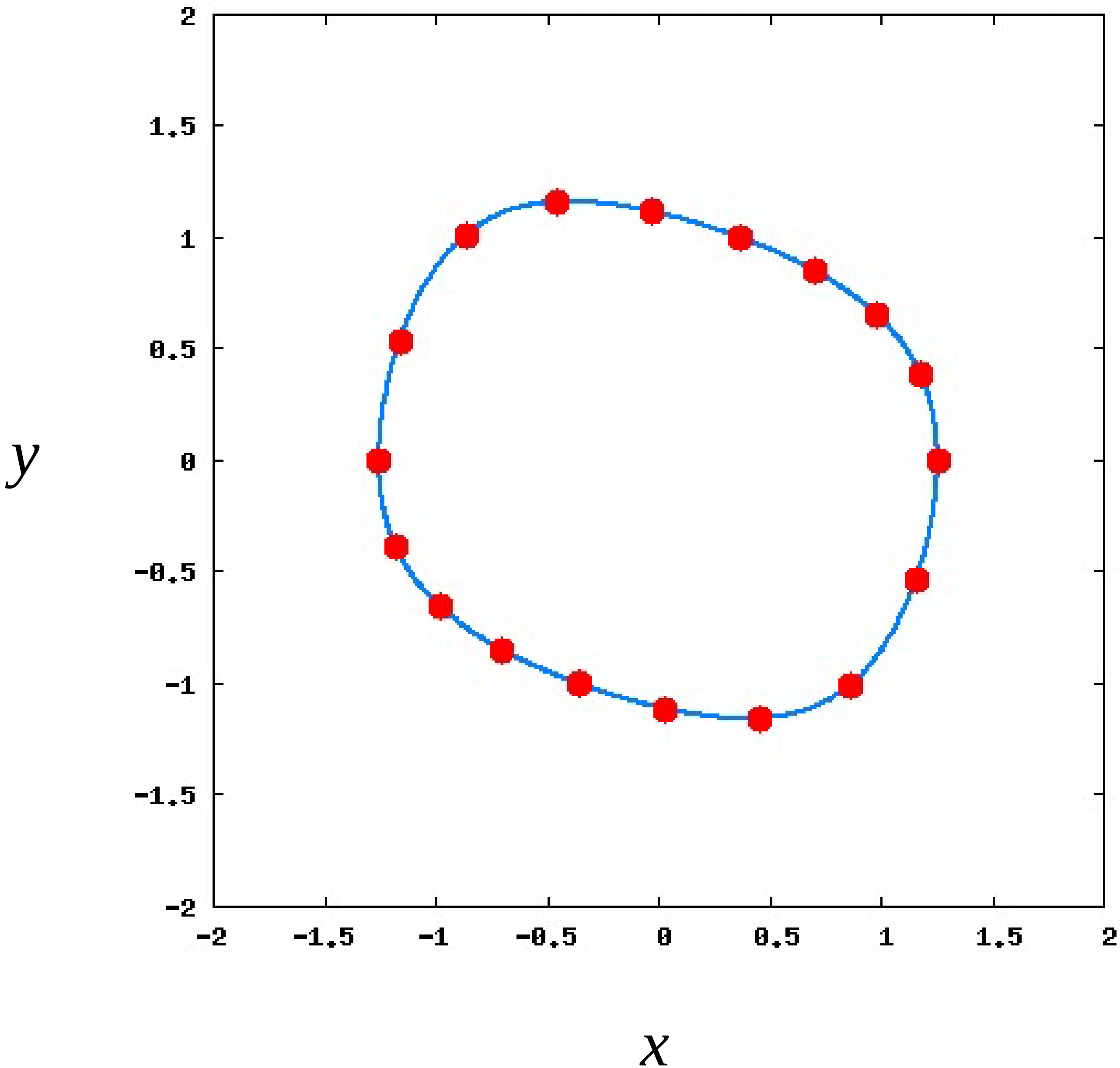
$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \alpha\left(\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - 1\right)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0$$

FitzHugh-Nagumo方程式

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{\varepsilon}\left(x - x^3 - y\right) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x - y + b \end{cases}$$

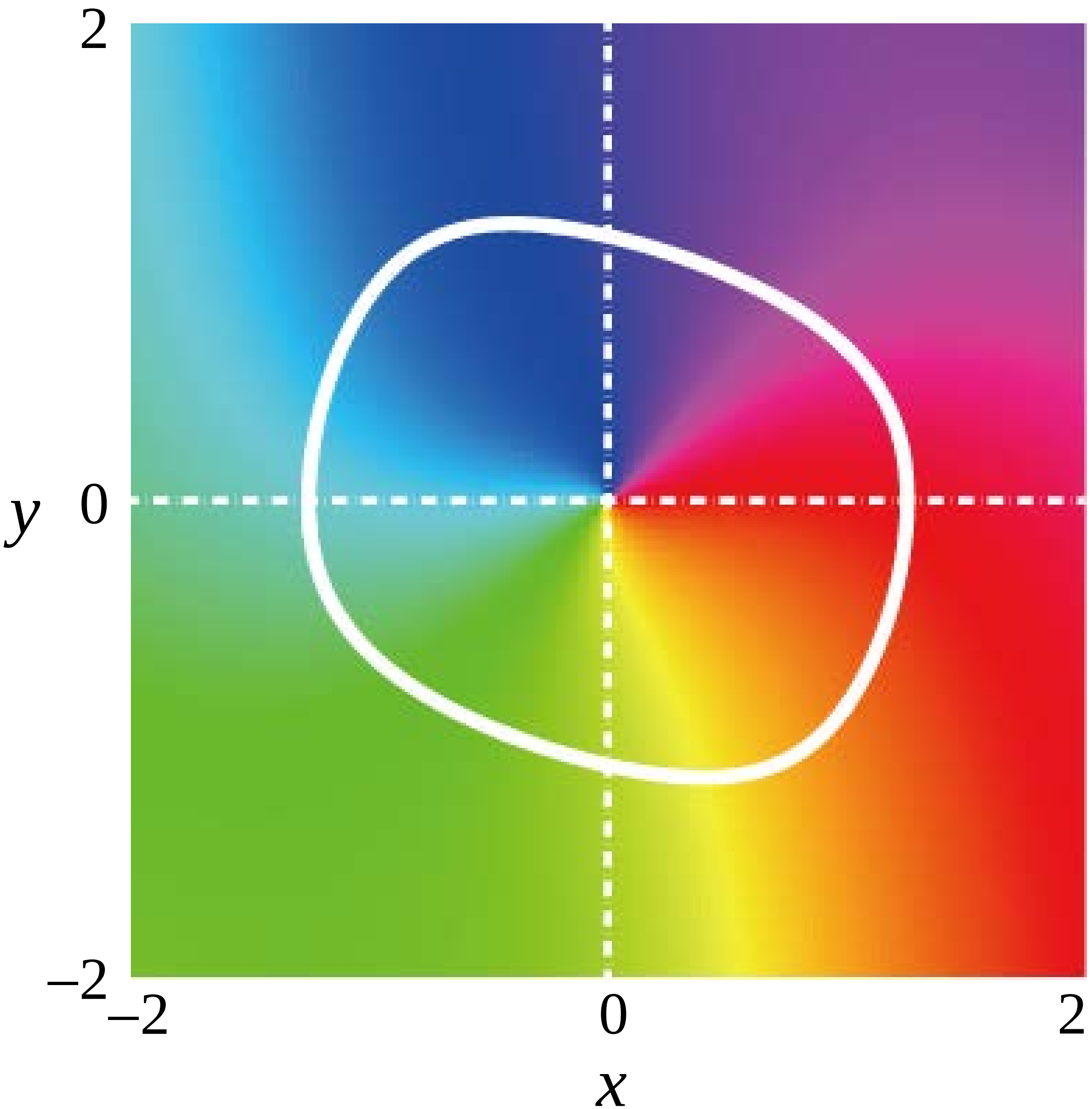
Rayleigh方程式



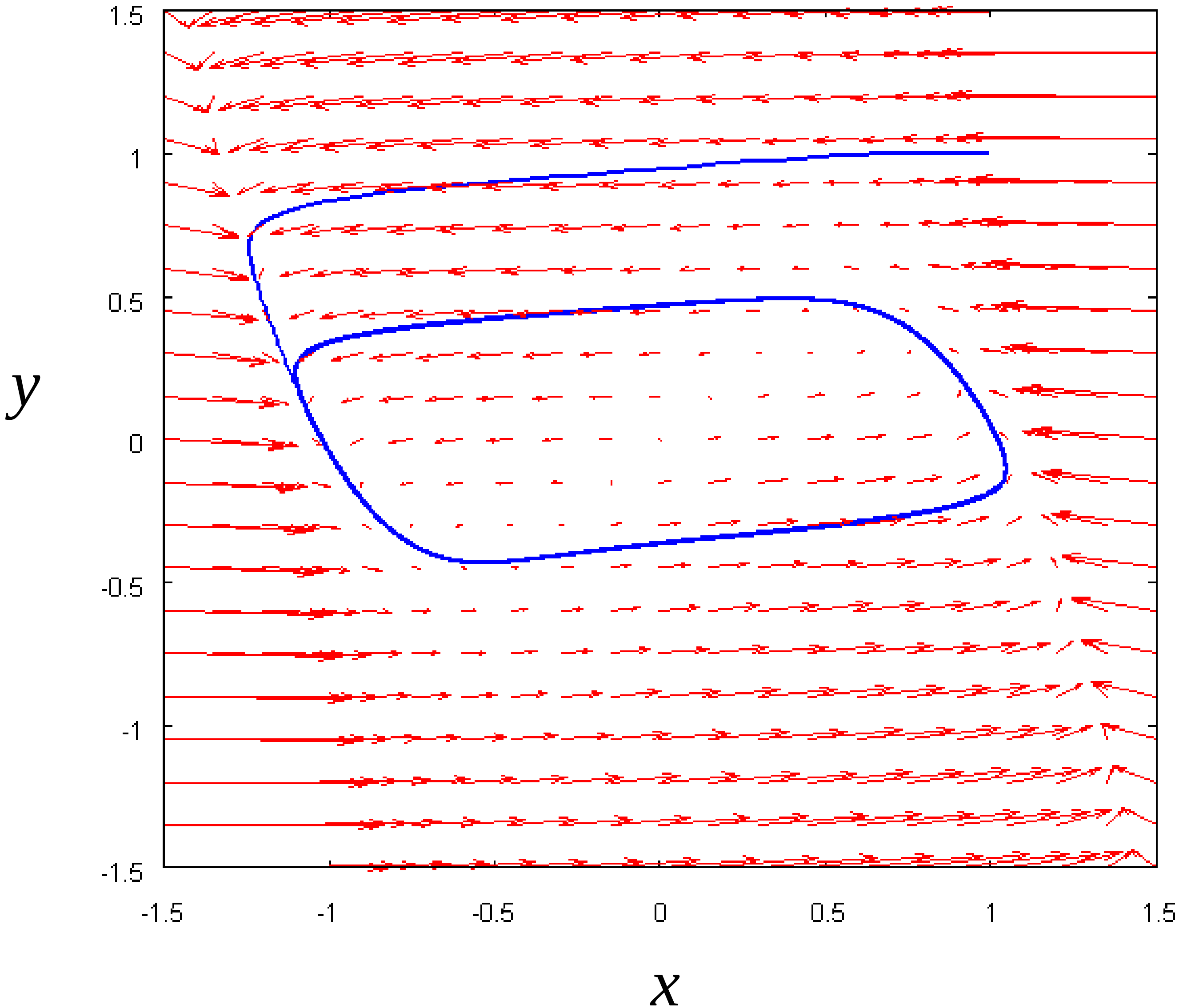


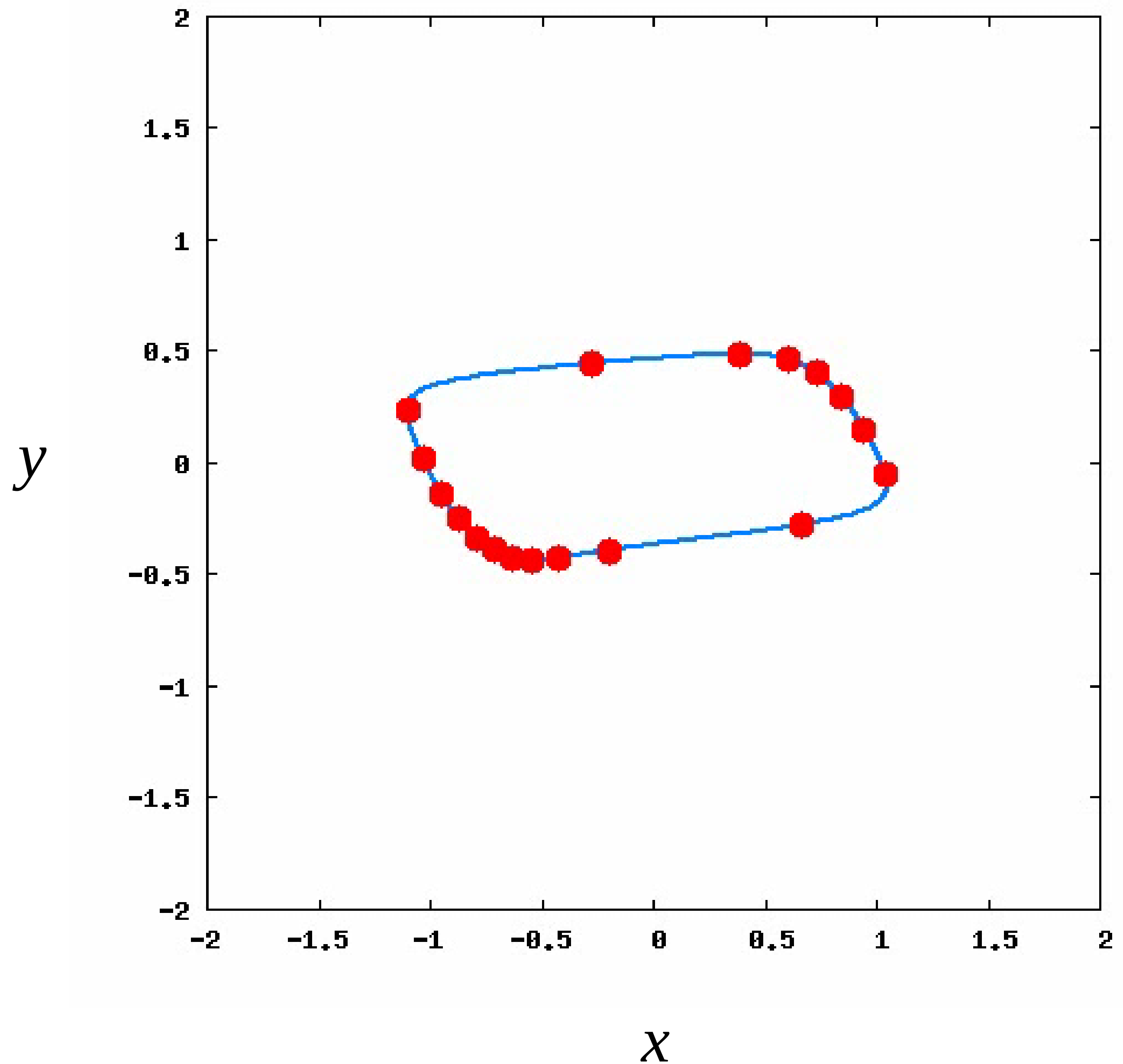
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
y
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x

等位相面



FitzHugh-Nagumo方程式





2
1.5
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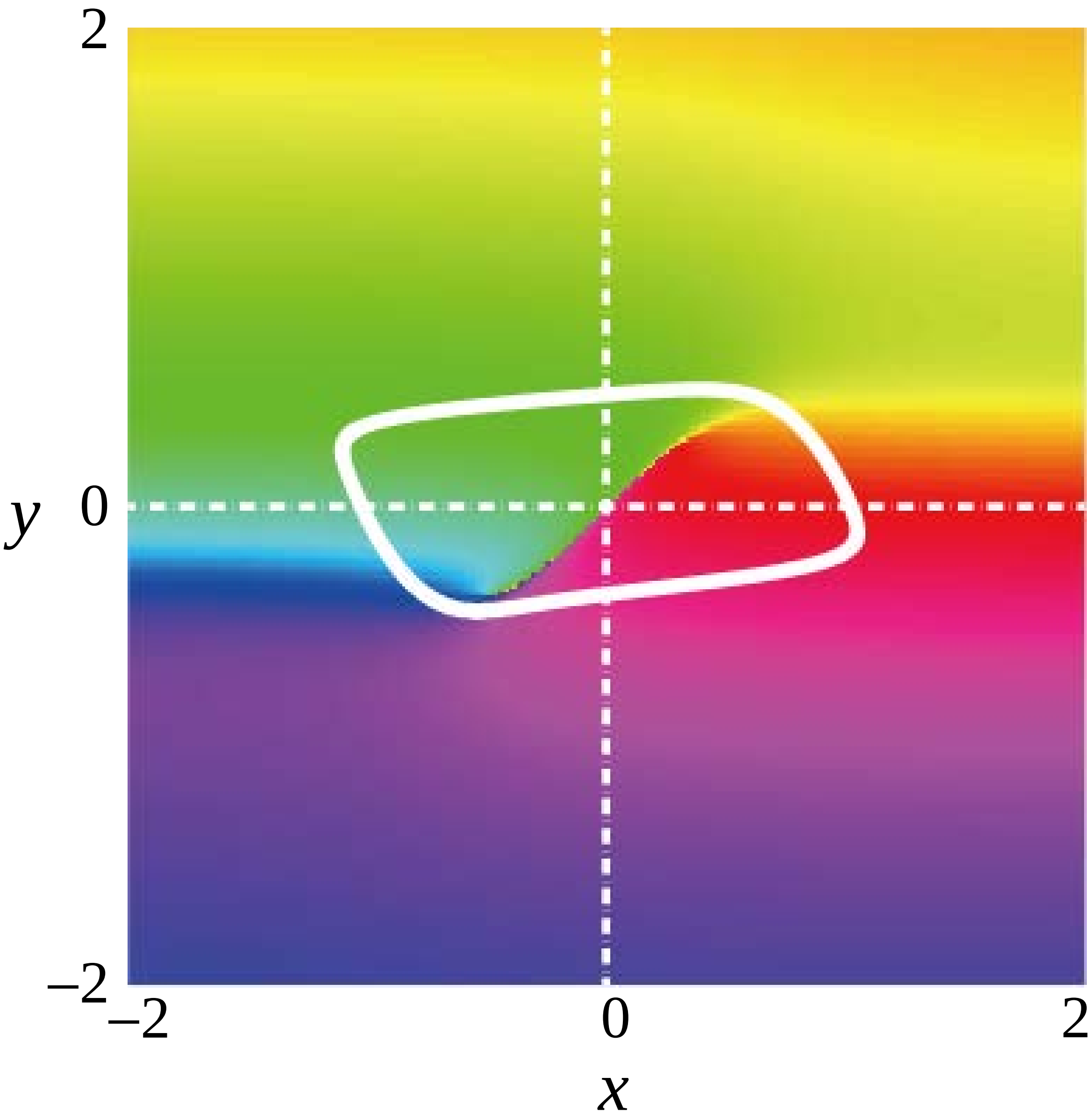
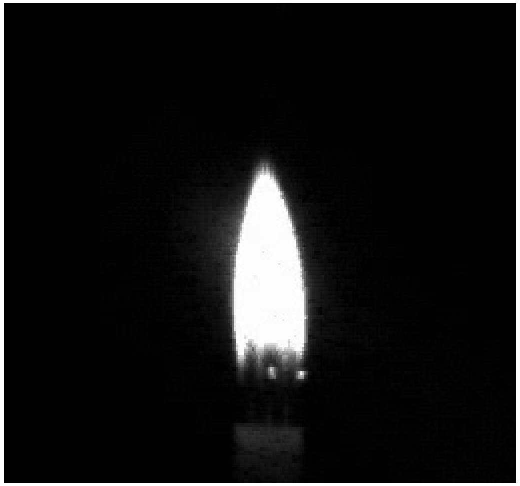
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
y
x

等位相面



実験室で見られる時空間秩序形成

さまざまなリズム現象

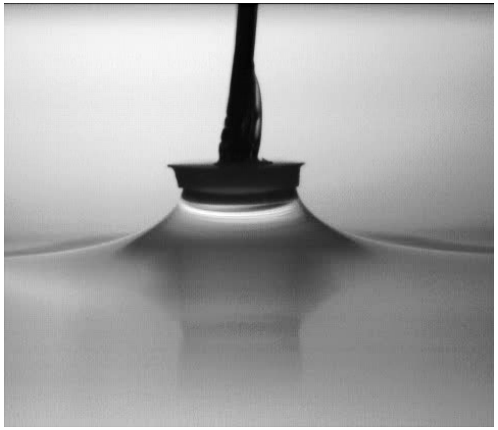


candle oscillator

BR reaction

saline oscillator

Plastic-bottle oscillator



water-camphor system

Belousov-Zhabotinsky (BZ)反応の実験

攪拌した系で

1 cm

空間勾配はなし

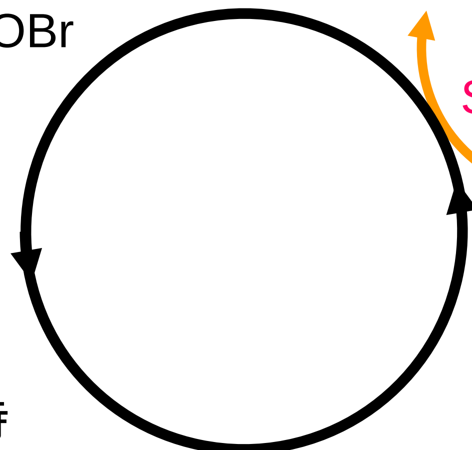
BZ反応のメカニズム

Step I. [Br $^-$]高の時

$Br^- + HBrO_2 \rightarrow 2HOBr$

($[HBrO_2]$と$[Br^-]$減)

抑制因子(Br^-)



Step III.

金属触媒の還元

（臭化マロン酸とマロン酸による）

マロン酸 ⟶ 臭化マロン酸

Step II. [Br $^-$]低の時

金属触媒の酸化

活性因子($HBrO_2$)を自己触媒的に生成

$\underline{HBrO_2} + BrO_3^-$

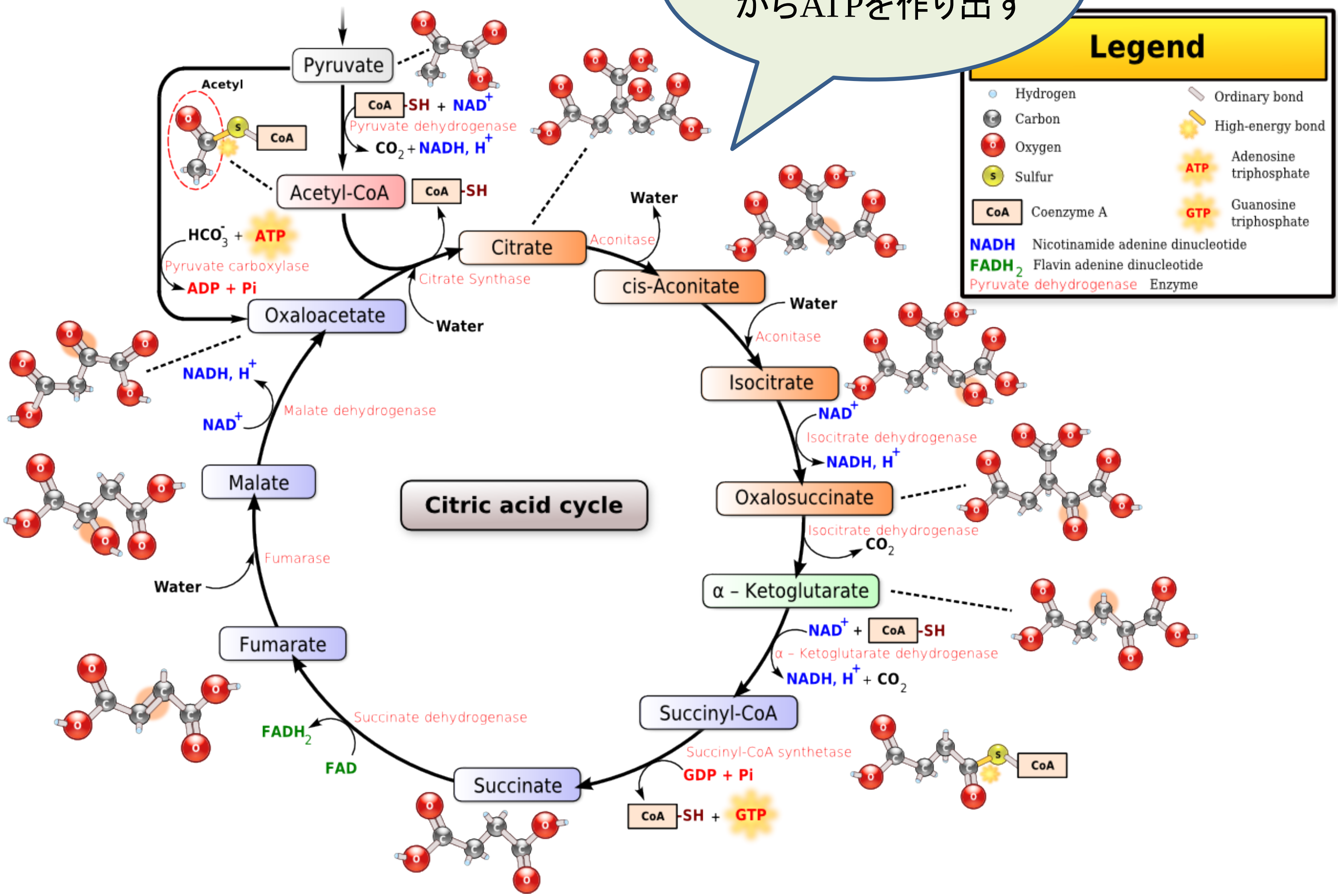
$+ 2[Fe(phen)_3]^{2+} + 3H^+$

$\rightarrow \underline{2HBrO_2} + 2[Fe(phen)_3]^{3+}$

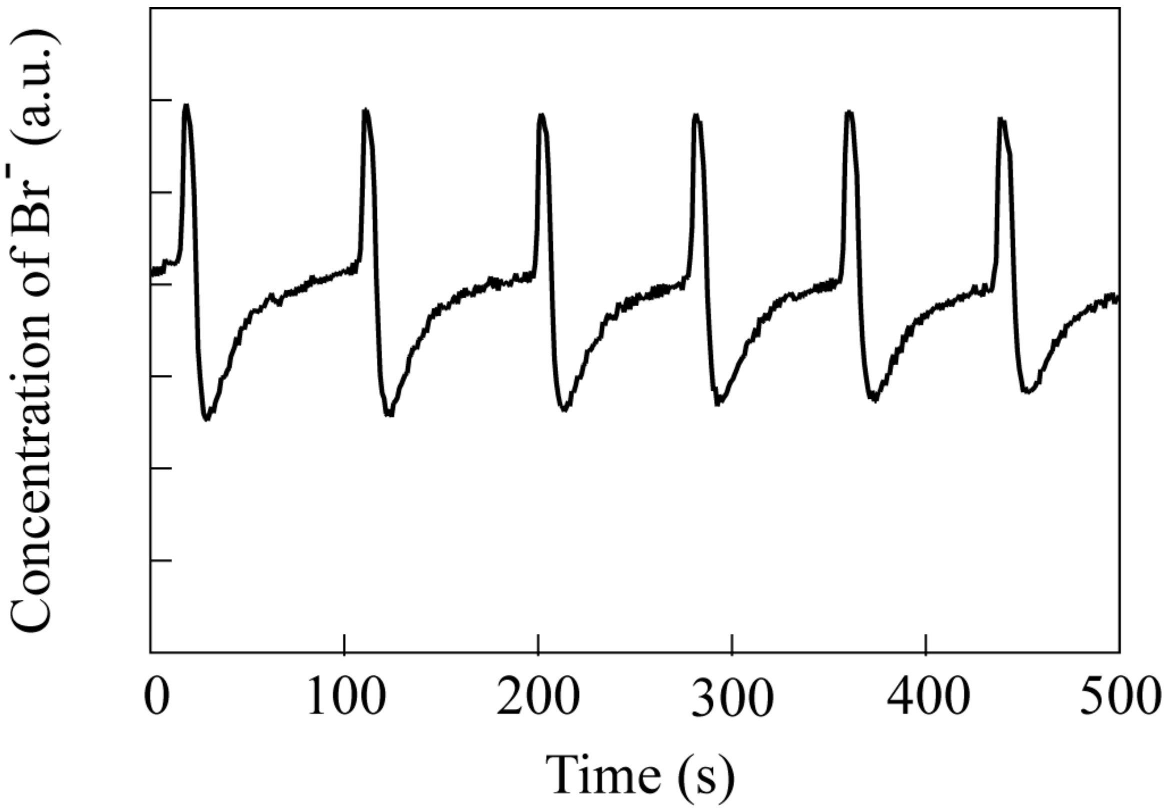
酸化反応　と　還元反応　が交互に起こる

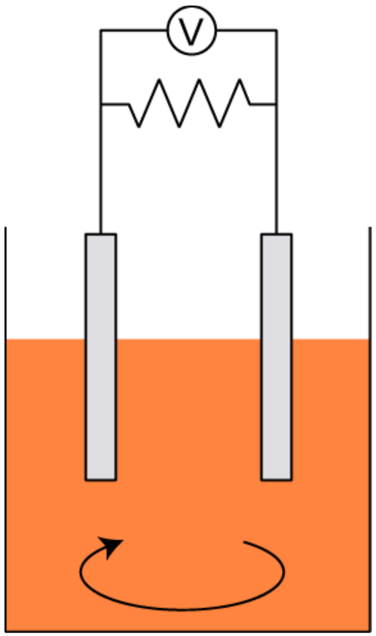
クエン酸回路

ブドウ糖
（正確にはピルビン酸）
からATPを作り出す



電極を用いた臭化物イオン（Br⁻）濃度の測定





BZ反応のモデル化

FKN Model (R. J. Field, E. Körös, and R. M. Noyes, 1972).

化学反応の素過程から重要なものを抽出

(R1) $Br^- + HOBr + H^+ \rightleftarrows Br_2 + H_2O$

(R2) $Br^- + HBrO_2 + H^+ \rightleftarrows 2HOBr$

(R3) $Br^- + BrO_3^- + 2H^+ \rightleftarrows HOBr + HBrO_2$

(R4) $2HBrO_2 \rightleftarrows HOBr + BrO_3^- + H^+$

(R5) $HBrO_2 + BrO_3^- + H^+ \rightleftarrows 2BrO_2\cdot + H_2O$

(R6) $BrO_2\cdot + Ce^{3+} + H^+ \rightleftarrows HBrO_2 + Ce^{4+}$

(R7) $BrO_2\cdot + Ce^{4+} + H_2O \rightleftarrows BrO_3^- + Ce^{3+} + 2H^+$

(R8) $Br_2 + CH_2(COOH)_2 \longrightarrow BrCH(COOH)_2 + Br^- + H^+$

(R9) $6Ce^{4+} + CH_2(COOH)_2 + 2H_2O \longrightarrow 6Ce^{3+} + HCOOH + 3CO_2 + 6H^+$

(R10) $4Ce^{4+} + BrCH(COOH)_2 + 2H_2O \longrightarrow 4Ce^{3+} + HCOOH + Br^- + 2CO_2 + 5H^+$

3変数Oregonator (R. J. Field and R. M. Noyes, 1974).

(R3) $BrO_3^- + Br^- + 2H^+ \longrightarrow HBrO_2 + HOBr$

(R2) $HBrO_2 + Br^- + H^+ \longrightarrow 2HOBr$

(R5)+2(R6) $2Ce^{3+} + BrO_3^- + HBrO_2 + 3H^+ \longrightarrow 2Ce^{4+} + 2HBrO_2 + H_2O$

(R4) $2HBrO_2 \longrightarrow BrO_3^- + HOBr + H^+$

(R10) $4Ce^{4+} + BrCH(COOH)_2 + 2H_2O \longrightarrow 4Ce^{3+} + HCOOH + Br^- + 2CO_2 + 5H^+$

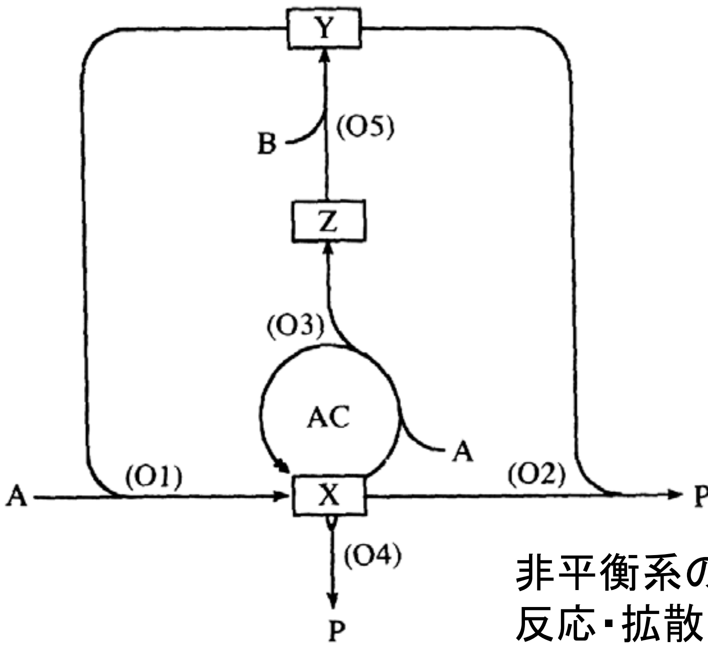
(O1) $A + Y \rightarrow X + P$

(O2) $X + Y \rightarrow 2P$

(O3) $A + X \rightarrow 2X + 2Z$

(O4) $2X \rightarrow A + P$

(O5) $Z + B \rightarrow hY$



非平衡系の科学III
反応・拡散系のダイナミクス
三池 秀敏ら 著

質量作用の法則より

$$\frac{\mathrm{d}X}{\mathrm{d}t} = k_3 H^2 AY - k_2 HXY + k_5 HAX - 2k_4 X^2$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = -k_3 H^2 AY - k_2 HXY + hk_j BZ$$

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = -2k_5 HAX - k_j BZ$$

化学反応には、速い反応と遅い反応がある。

速い反応は、すぐに平衡に達すると考えて、
近似することができる。

＝ 断熱消去

例:　$\mathrm{L} \underset{\text{速}}{\longrightarrow} \mathrm{M} \underset{\text{遅}}{\longrightarrow} \mathrm{N}$

このとき、Nの濃度の時間変化はLの濃度のみによると
考えることができる。

化学反応の素過程

↓

FKN Model (R. J. Field, E. Körös, and R. M. Noyes, 1972).

↓

3変数Oregonator (R. J. Field and R. M. Noyes, 1974).

↓

2変数Oregonator (J. J. Tyson and P. C. Fife, 1980).

$$\frac{\mathrm{d}U}{\mathrm{d}t} = f(U,V)$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = g(U,V)$$

$$f(U,V) = \frac{1}{\varepsilon}\left(U(1-U) - fV\frac{U-q}{U+q}\right)$$

$$g(U,V) = U - V$$

U : $[HBrO_2]$

V : $[Fe(phen)_3^{3+}]$

青色

2変数Oregonator

(J. J. Tyson and P. C. Fife, 1980).

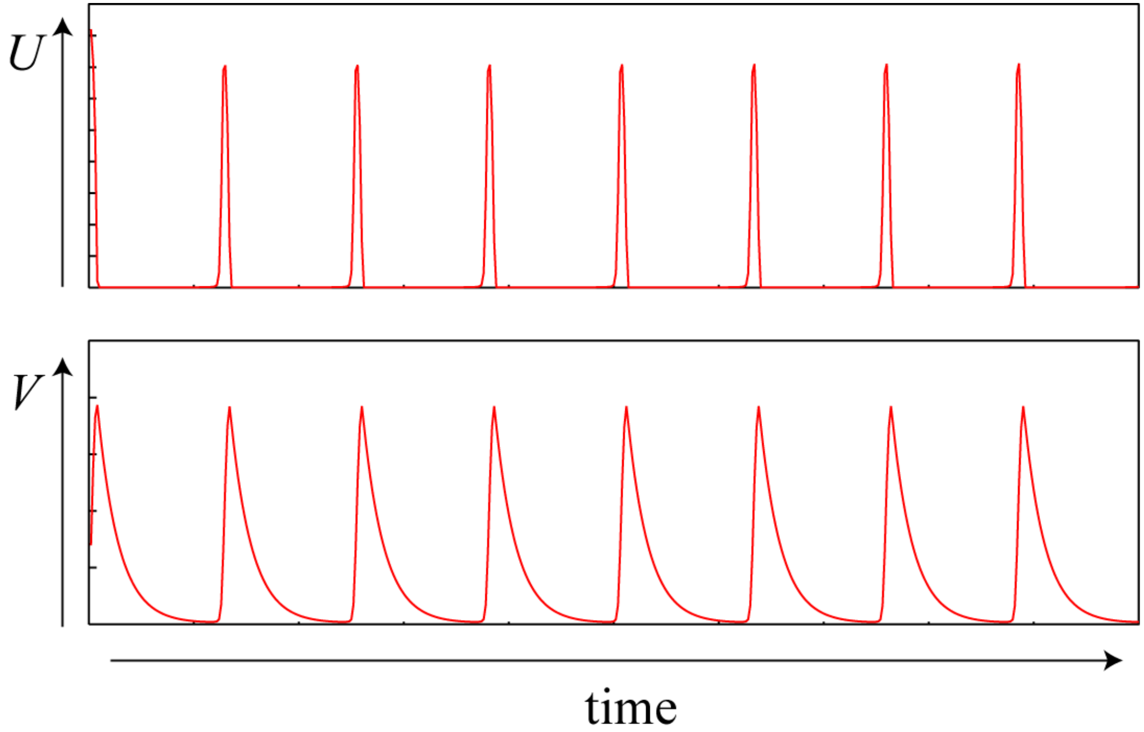
$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{1}{\varepsilon}\left(U(1-U) - fV\frac{U-q}{U+q} \right)$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = U - V$$

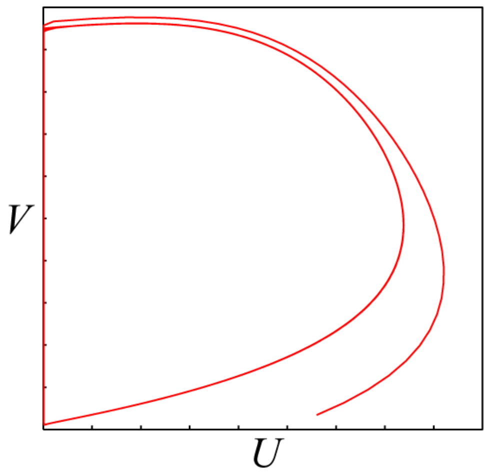
U : [$HBrO_2$]

V : [$Fe(phen)_3^{3+}$]　　青色

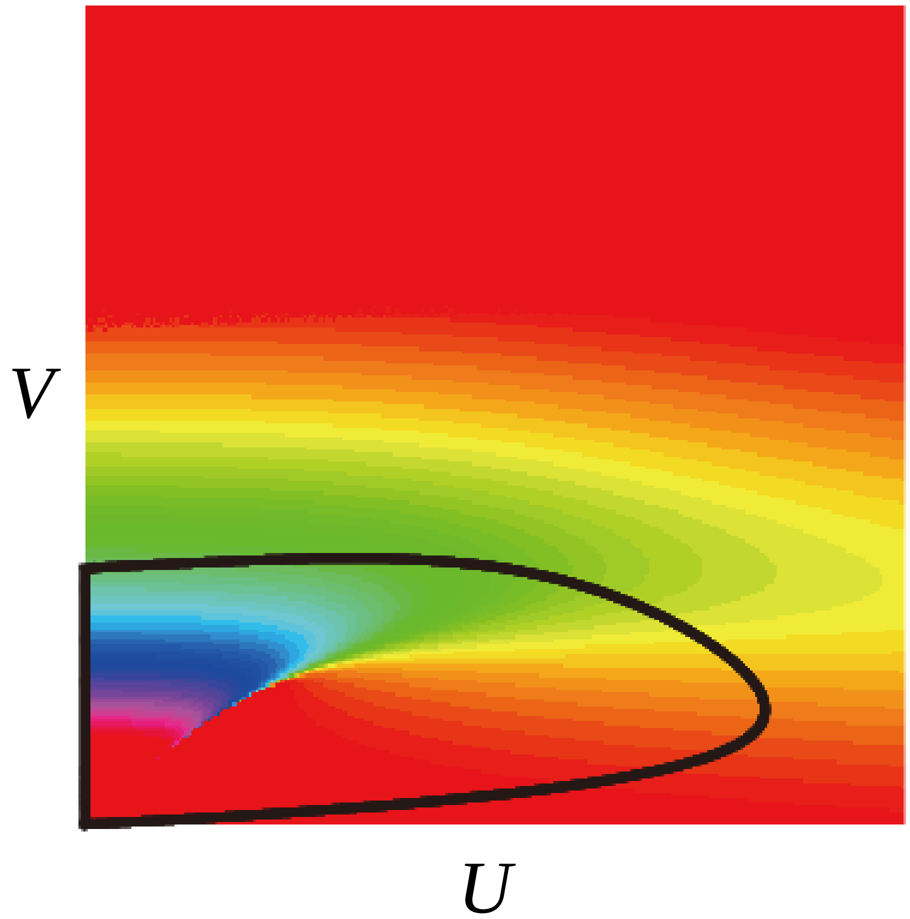
数値計算:



相平面で見ると

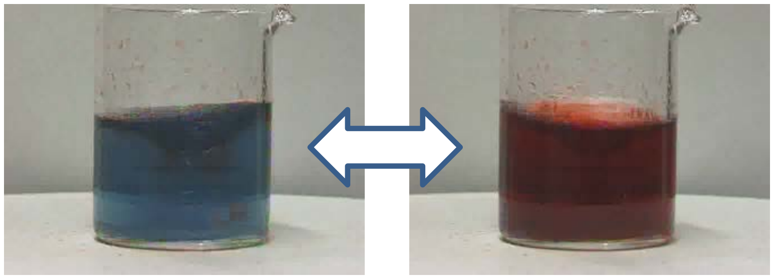


Oregonatorの等位相面



どちらもリミットサイクル振動なので、「位相」で考えられる。

具体(現象)から抽象(理論)へ



BZ反応

化学反応式
(物質に近いモデル)

(R1) $Br^- + HOBr + H^+ \rightleftarrows Br_2 + H_2O$

(R2) $Br^- + HBrO_2 + H^+ \rightleftarrows 2HOBr$

(R3) $Br^- + BrO_3^- + 2H^+ \rightleftarrows HOBr + HBrO_2$

(R4) $2HBrO_2 \rightleftarrows HOBr + BrO_3^- + H^+$

(R5) $HBrO_2 + BrO_3^- + H^+ \rightleftarrows 2BrO_2\cdot + H_2O$

(R6) $BrO_2\cdot + Ce^{3+} + H^+ \rightleftarrows HBrO_2 + Ce^{4+}$

(R7) $BrO_2\cdot + Ce^{4+} + H_2O \rightleftarrows BrO_3^- + Ce^{3+} + 2H^+$

(R8) $Br_2 + CH_2(COOH)_2 \longrightarrow BrCH(COOH)_2 + Br^- + H^+$

(R9) $6Ce^{4+} + CH_2(COOH)_2 + 2H_2O \longrightarrow 6Ce^{3+} + HCOOH + 3CO_2 + 6H^+$

(R10) $4Ce^{4+} + BrCH(COOH)_2 + 2H_2O \longrightarrow 4Ce^{3+} + HCOOH + Br^- + 2CO_2 + 5H^+$

Oregonator
(断熱近似・無次元化)

$$\frac{dU}{dt} = f(U,V)$$

$$\frac{dV}{dt} = g(U,V)$$

$$f(U,V) = \frac{1}{\varepsilon}\left(U(1-U) - fV\frac{U-q}{U+q}\right)$$

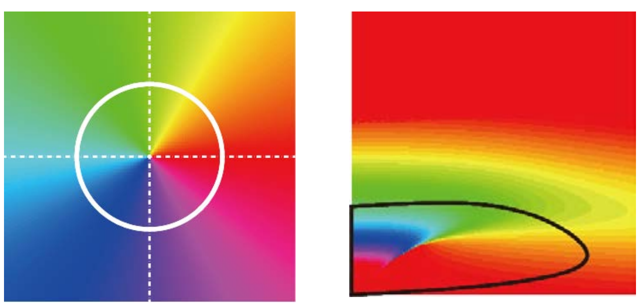
$$g(U,V) = U - V$$

U: $[HBrO_2]$

V: $[Fe(phen)_3^{3+}]$

青色

Stuart-Landau方程式
(位相記述・分岐理論)



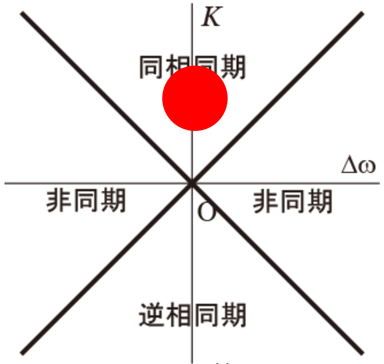
2014.1.13
物性物理学C

非線形振動子の結合系

非線形振動子の結合系

2振動子の結合系

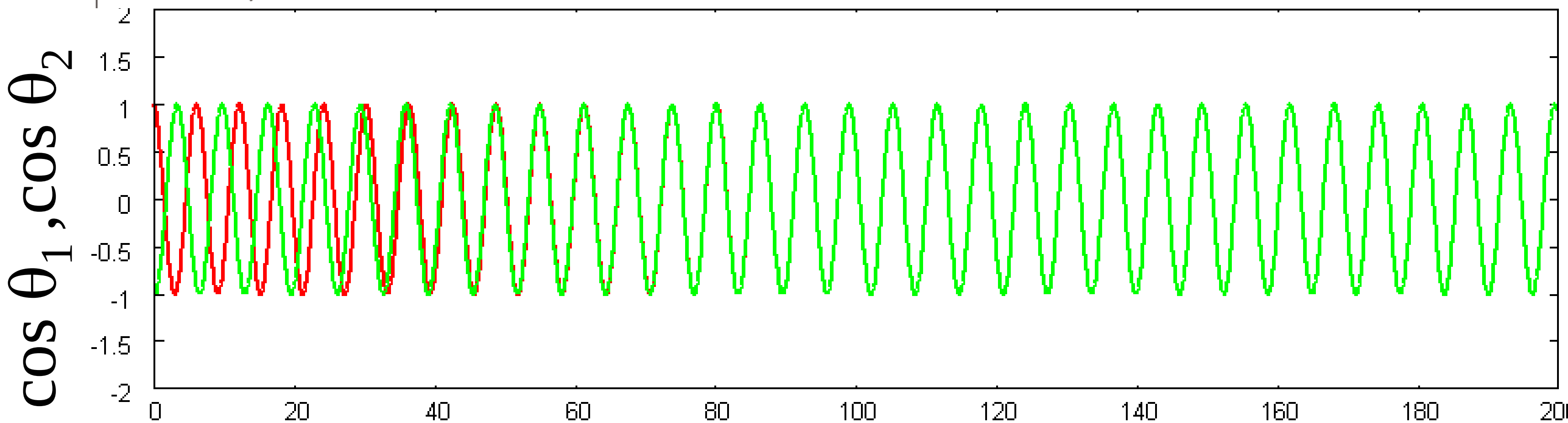
$$\begin{cases} \dfrac{\mathrm{d}\theta_1}{\mathrm{d}t} = \omega_1 + K\sin(\theta_2 - \theta_1) \\ \dfrac{\mathrm{d}\theta_2}{\mathrm{d}t} = \omega_2 + K\sin(\theta_1 - \theta_2) \end{cases}$$

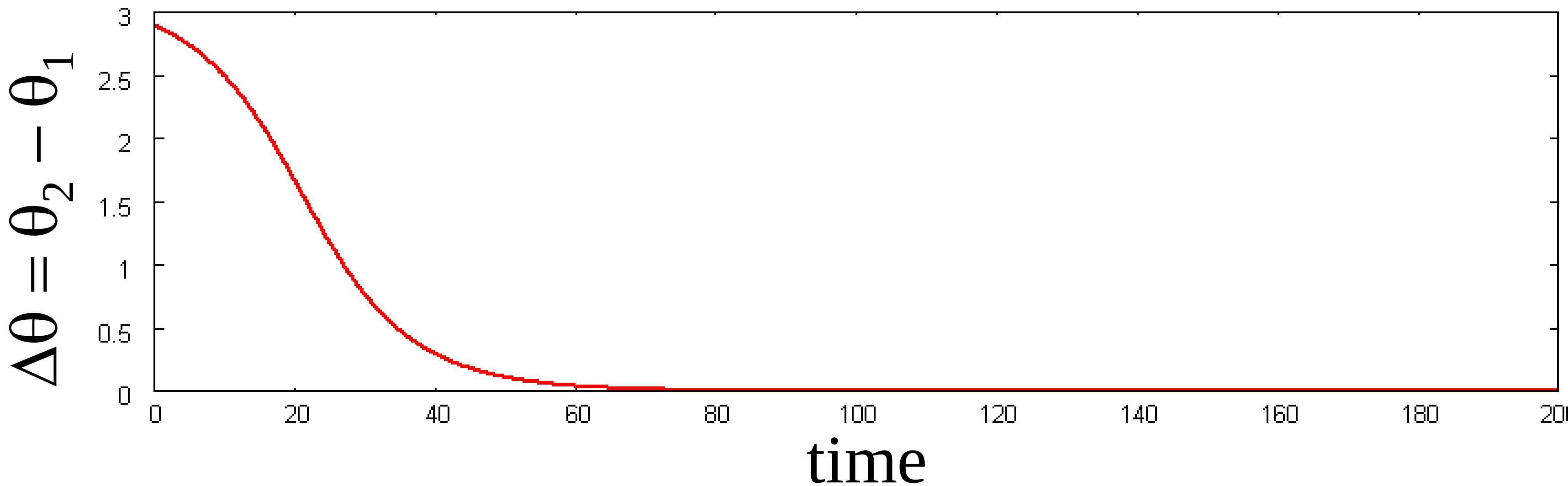


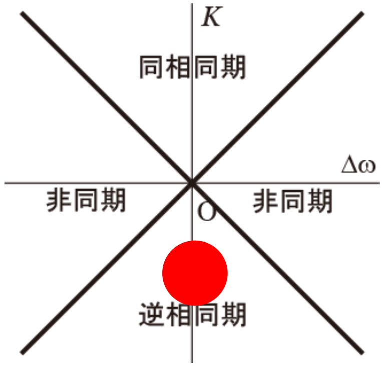
$$
\begin{cases}
\dfrac{\mathrm{d}\theta_1}{\mathrm{d}t} = \omega_1 + K\sin(\theta_2 - \theta_1) \\
\dfrac{\mathrm{d}\theta_2}{\mathrm{d}t} = \omega_2 + K\sin(\theta_1 - \theta_2)
\end{cases}
$$

$\omega_1 = \omega_2 = 1$

$K = 0.05$



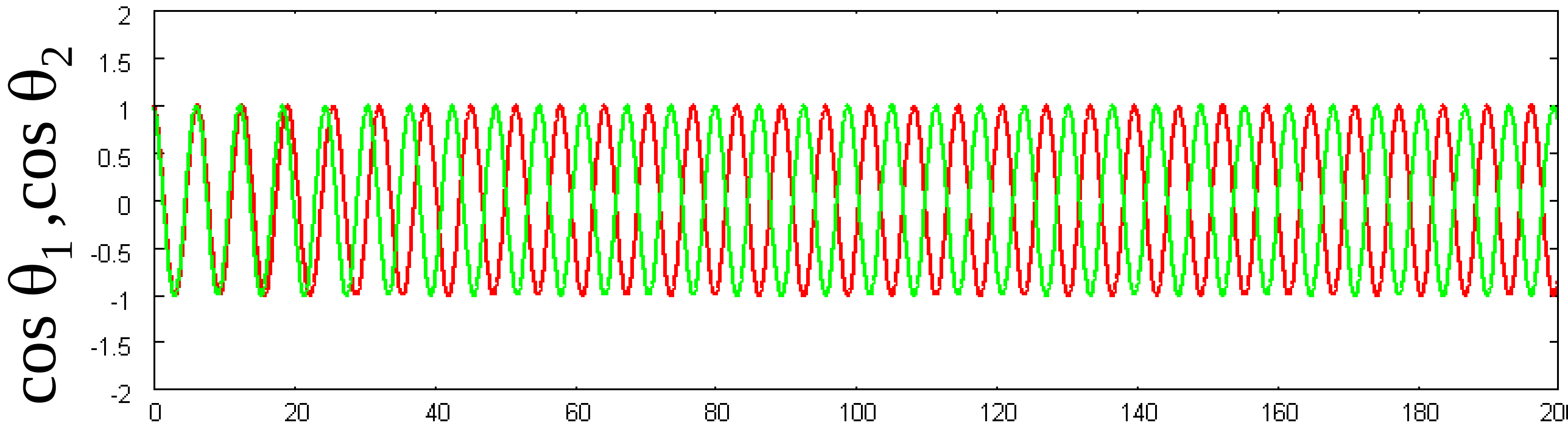


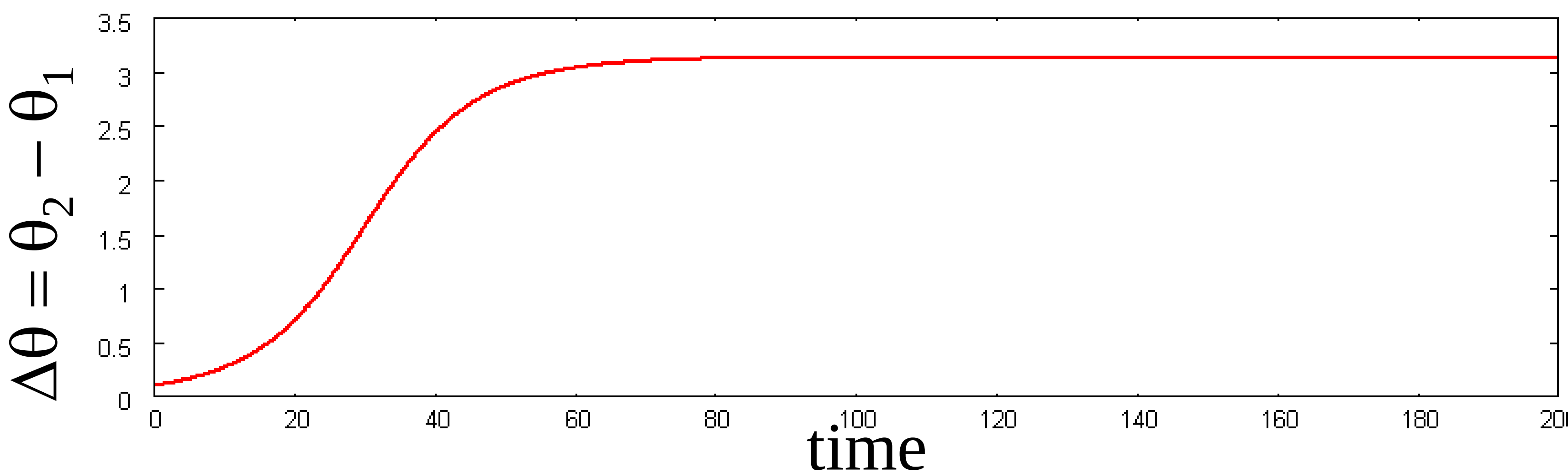


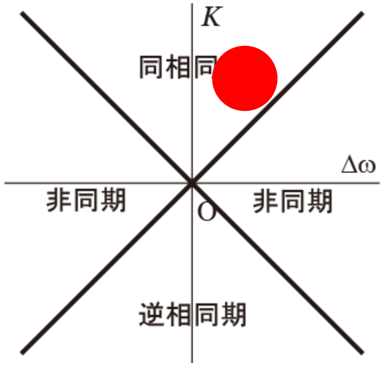
$$\begin{cases} \dfrac{d\theta_1}{dt} = \omega_1 + K\sin(\theta_2 - \theta_1) \\ \dfrac{d\theta_2}{dt} = \omega_2 + K\sin(\theta_1 - \theta_2) \end{cases}$$

$\omega_1 = \omega_2 = 1$

$K = -0.05$



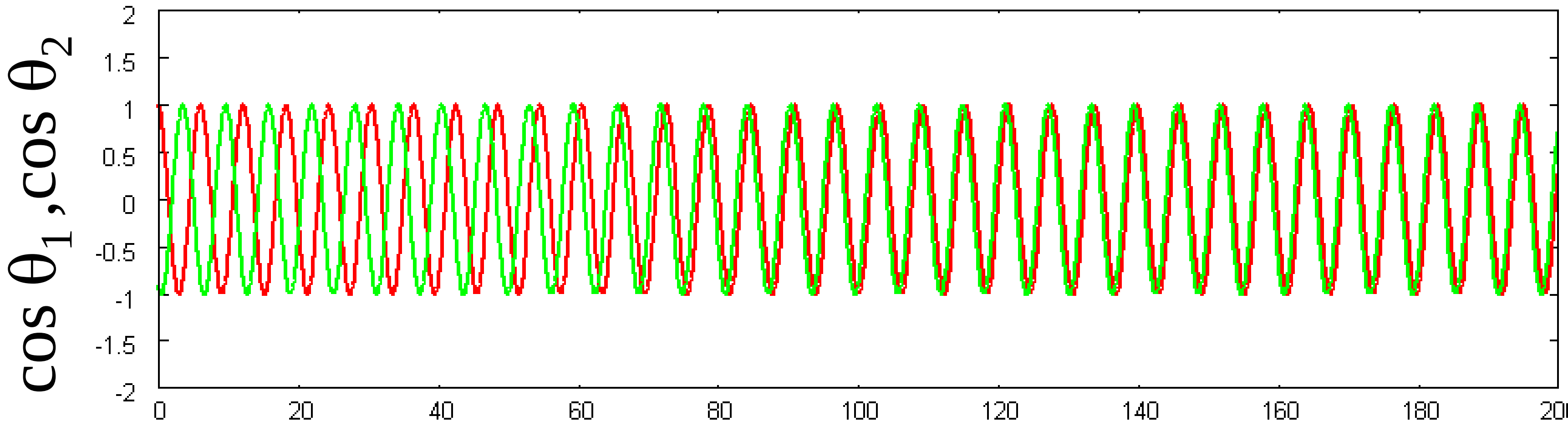


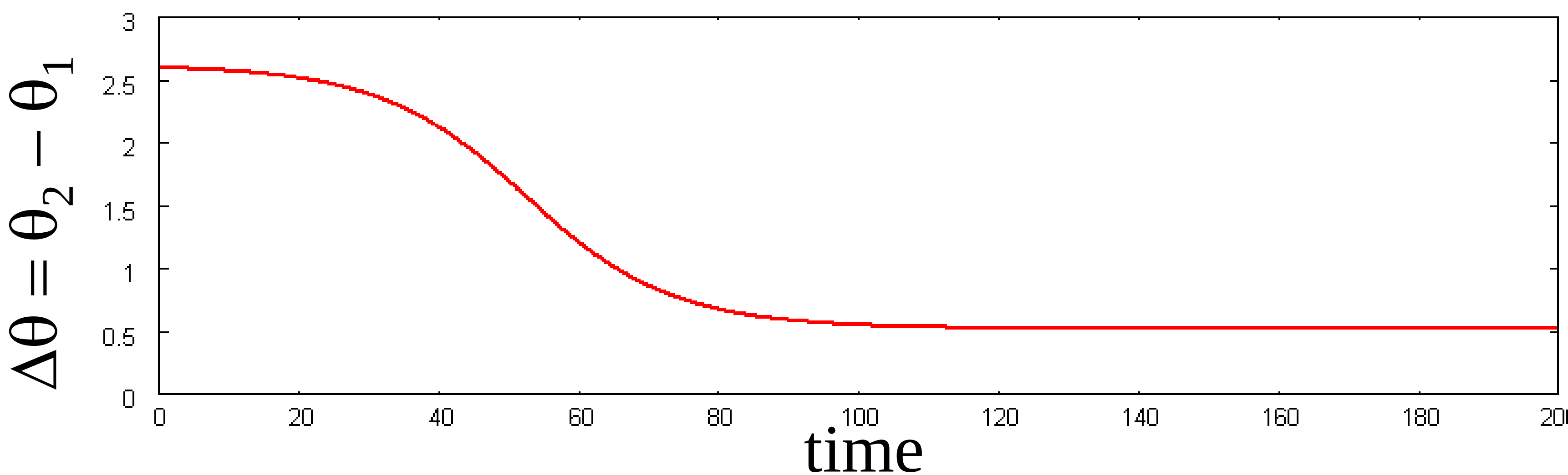


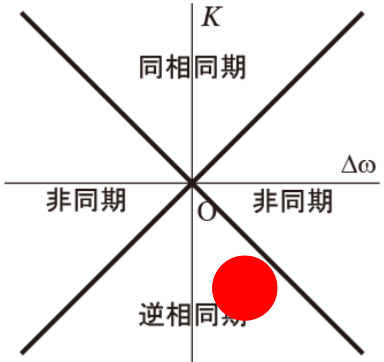
$$
\left\{
\begin{aligned}
\frac{\mathrm{d}\theta_1}{\mathrm{d}t} &= \omega_1 + K\sin(\theta_2 - \theta_1) \\
\frac{\mathrm{d}\theta_2}{\mathrm{d}t} &= \omega_2 + K\sin(\theta_1 - \theta_2)
\end{aligned}
\right.
$$

$\omega_1 = 1, \ \omega_2 = 1.05$

$K = 0.05$



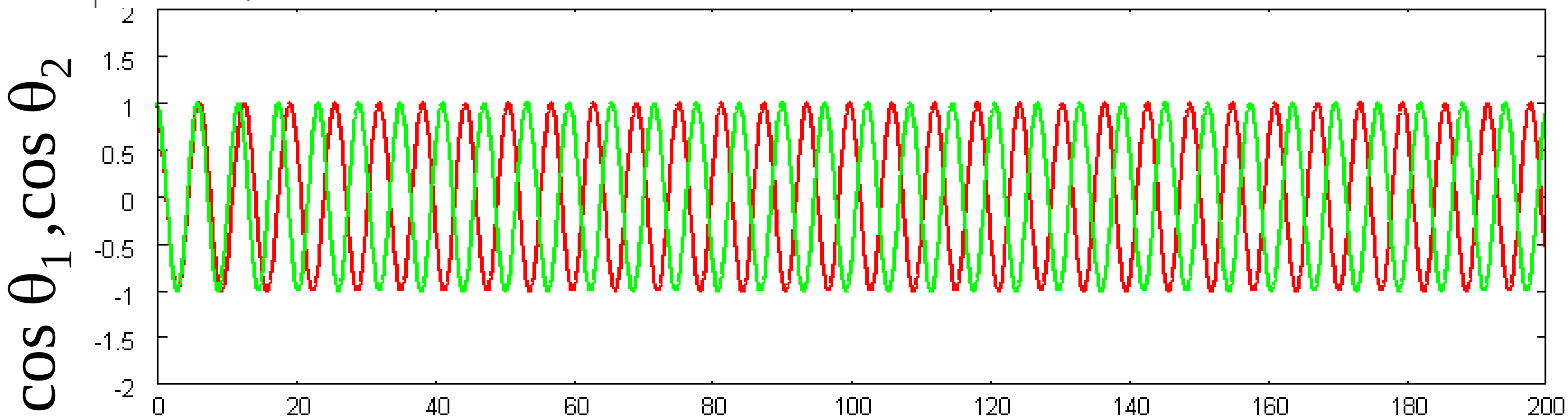


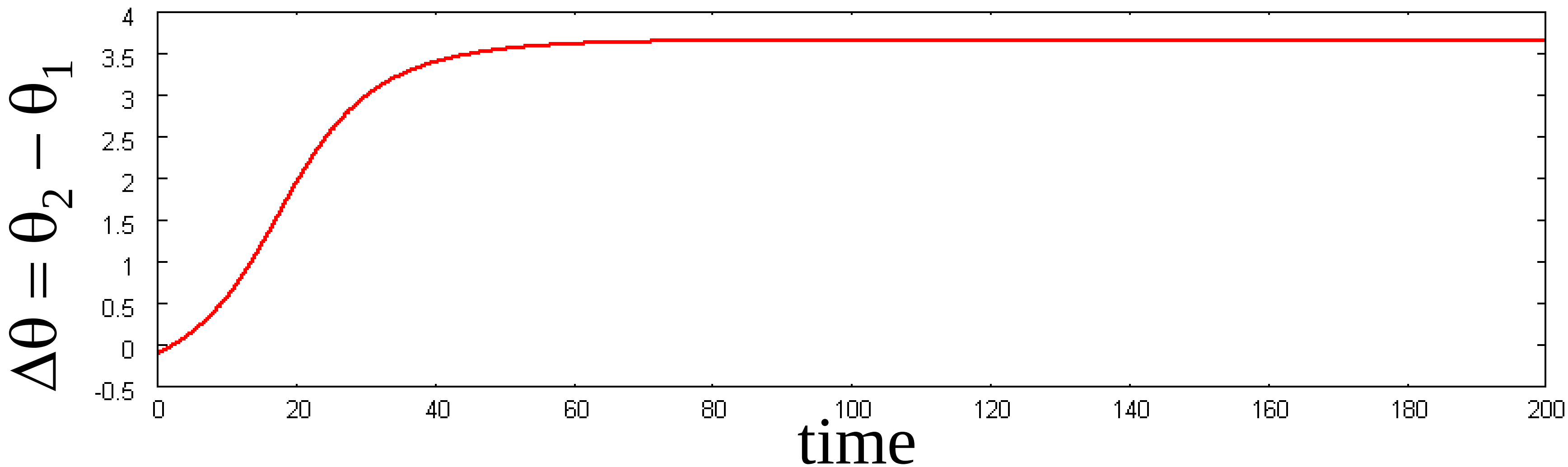


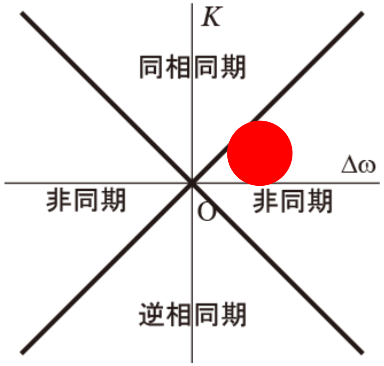
$$\begin{cases} \dfrac{\mathrm{d}\theta_1}{\mathrm{d}t} = \omega_1 + K\sin(\theta_2 - \theta_1) \\ \dfrac{\mathrm{d}\theta_2}{\mathrm{d}t} = \omega_2 + K\sin(\theta_1 - \theta_2) \end{cases}$$

$\omega_1 = 1,\ \omega_2 = 1.05$

$K = -0.05$



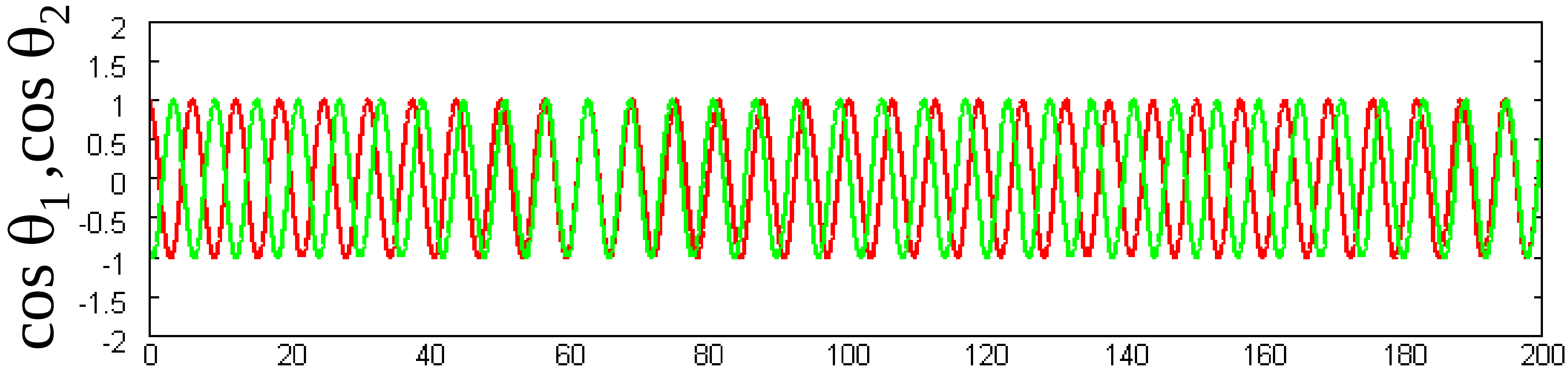


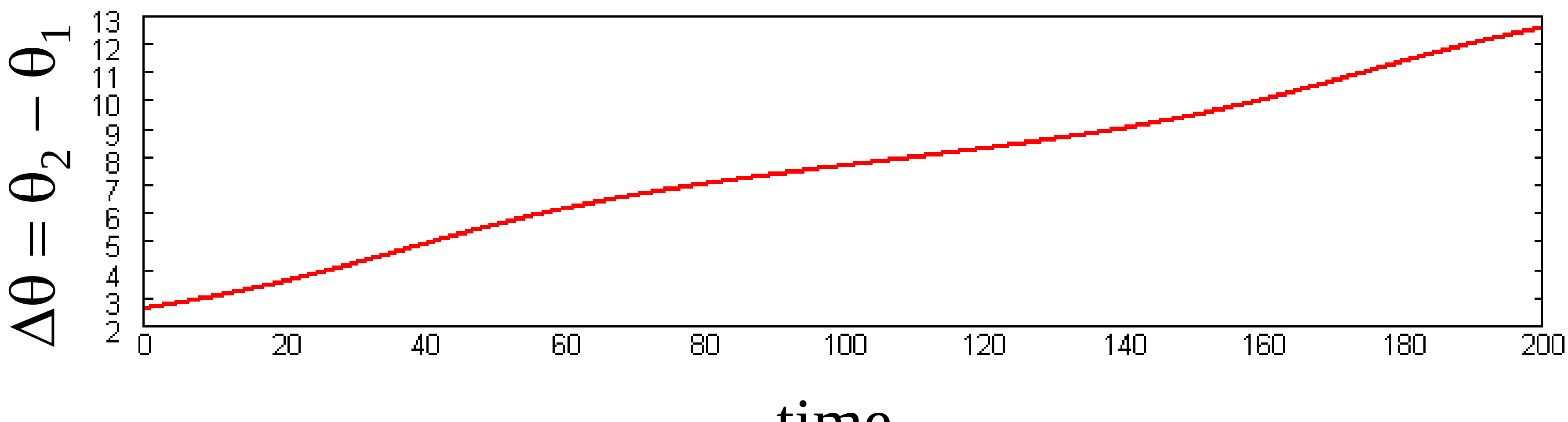


$$
\begin{cases}
\dfrac{d\theta_1}{dt} = \omega_1 + K\sin(\theta_2 - \theta_1) \\
\dfrac{d\theta_2}{dt} = \omega_2 + K\sin(\theta_1 - \theta_2)
\end{cases}
$$

$\omega_1 = 1,\ \omega_2 = 1.05$

$K = 0.01$





2015.1.13
物性物理学C

反応拡散系とパターン形成

Turingパターン（静止パターン）

$$\frac{\partial u}{\partial t} = -u^3 + u - 4v + D_u \nabla^2 u$$

$$\frac{\partial v}{\partial t} = u - 3v - a + D_v \nabla^2 v$$

$a = 0$では$u = v = 0$が固定点

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$$

tr A = –2、det A = 1より安定

$$u = 0 + \int \mathrm{d}k\ \Delta u(k) e^{ikx}$$

$$v = 0 + \int \mathrm{d}k\ \Delta v(k) e^{ikx}$$ とおいて

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} = \begin{pmatrix} 1 - D_u k^2 & -4 \\ 1 & -3 - D_u k^2 \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$$

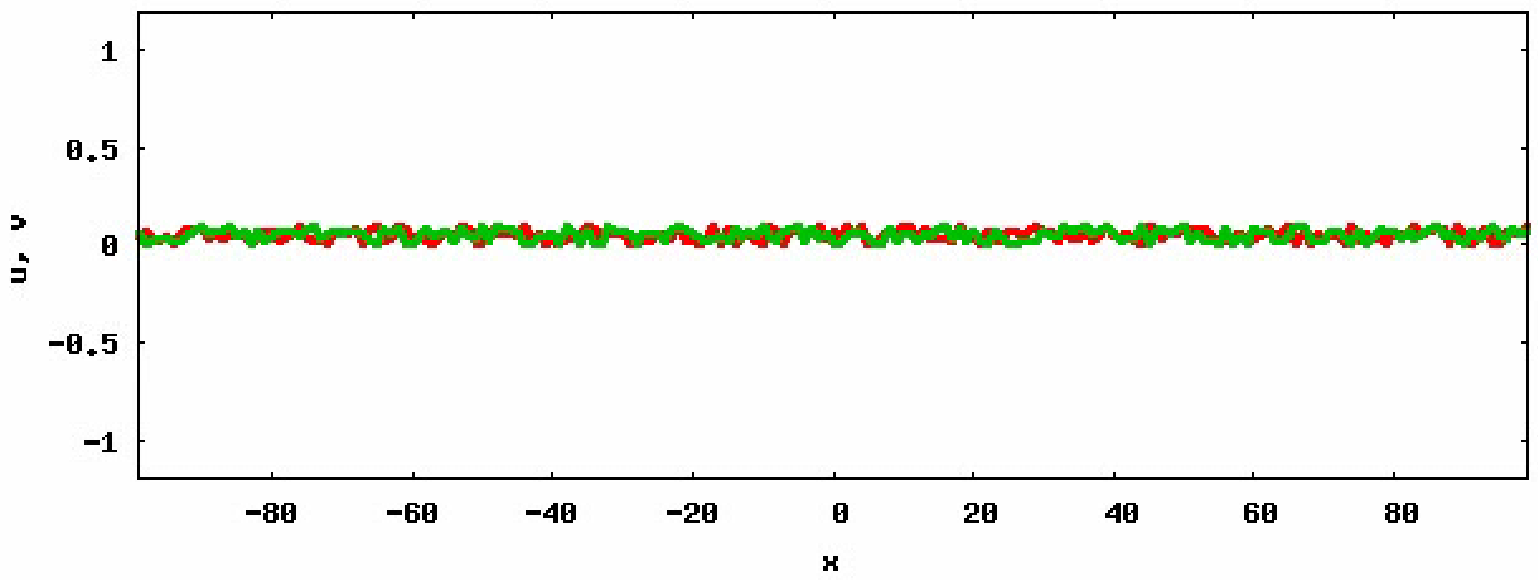
不安定化する条件は

$$D_v > 9D_u$$

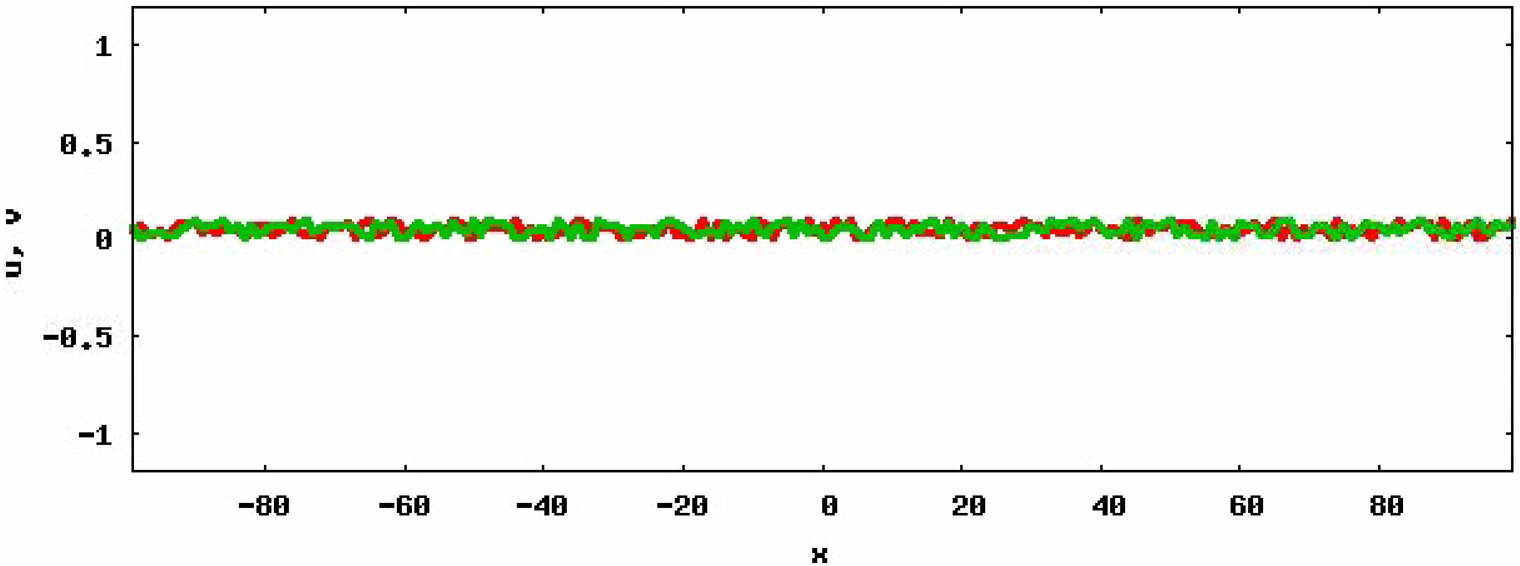
はじめに不安定化する波数は　$k = \sqrt{\frac{1}{2D_u} - \frac{3}{2D_v}}$

1次元では

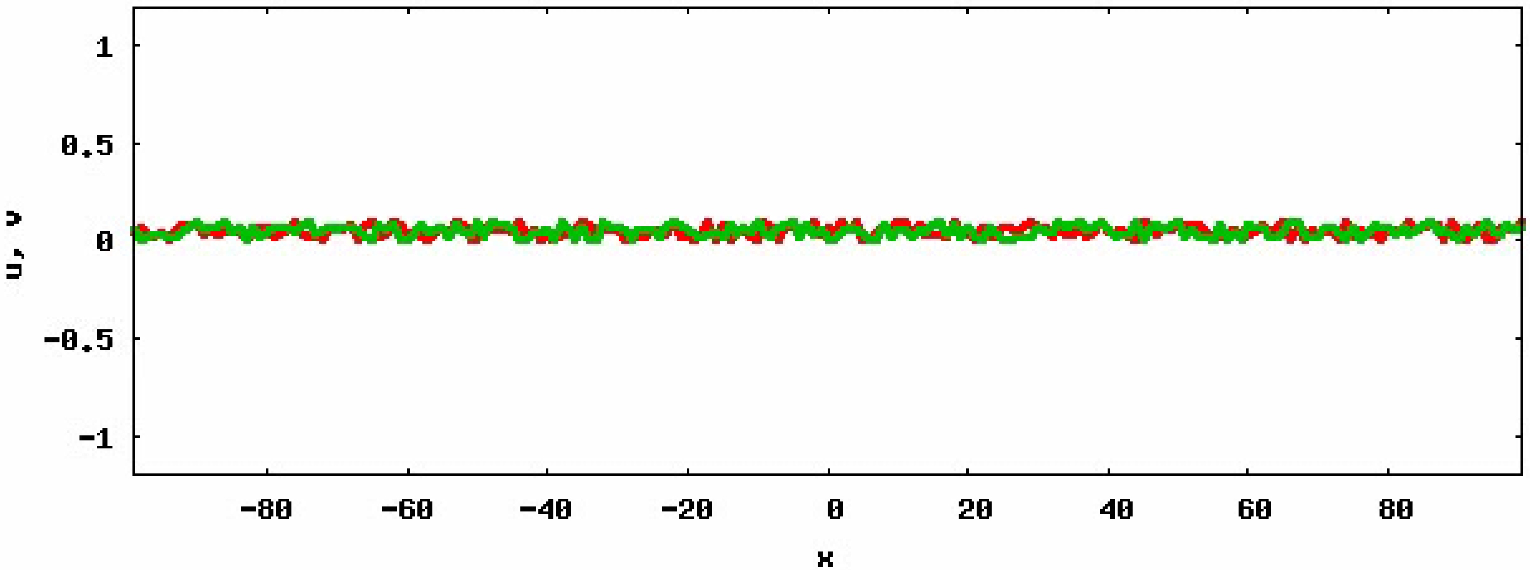
$$a = 0, \quad D_u = 1, D_v = 5$$



$a = 0, \quad D_u = 1, D_v = 20$

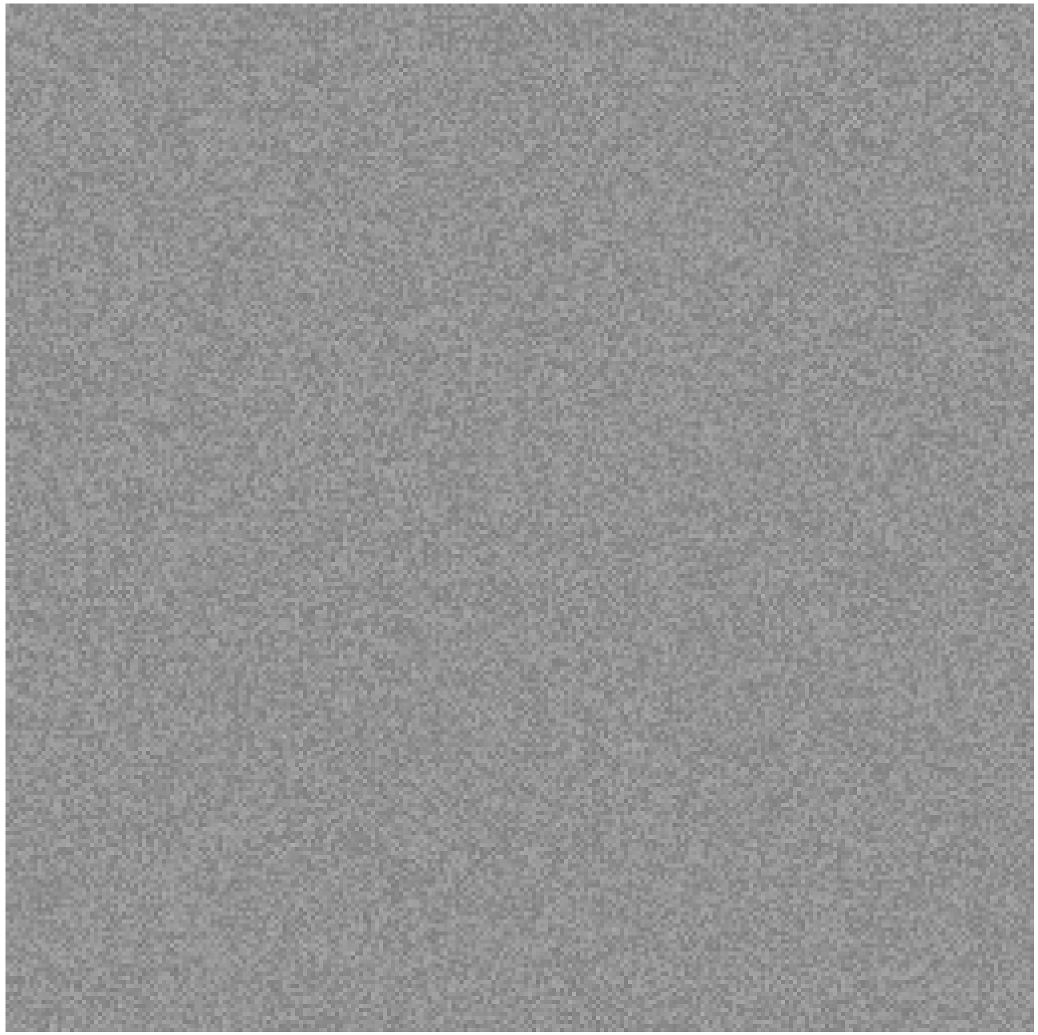


$a = 0, \quad D_u = 2, D_v = 20$

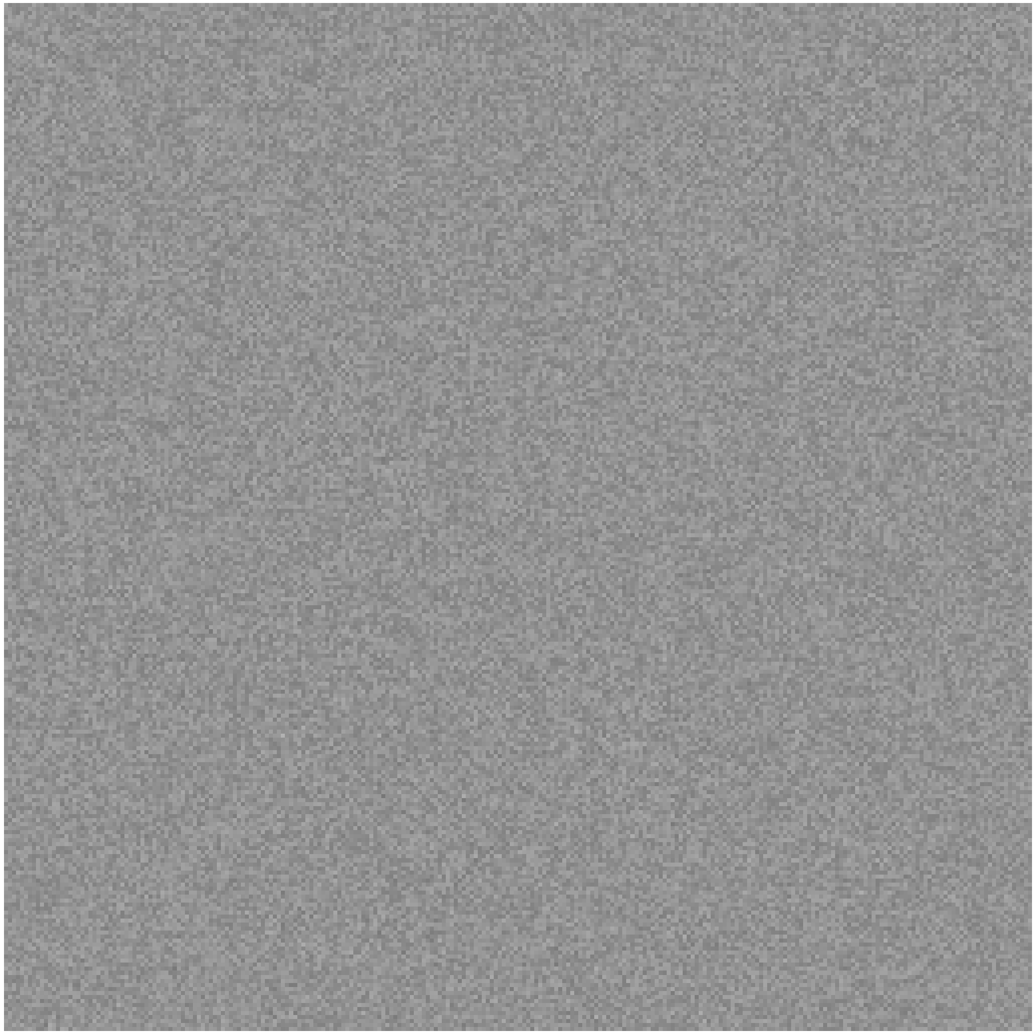


2次元では

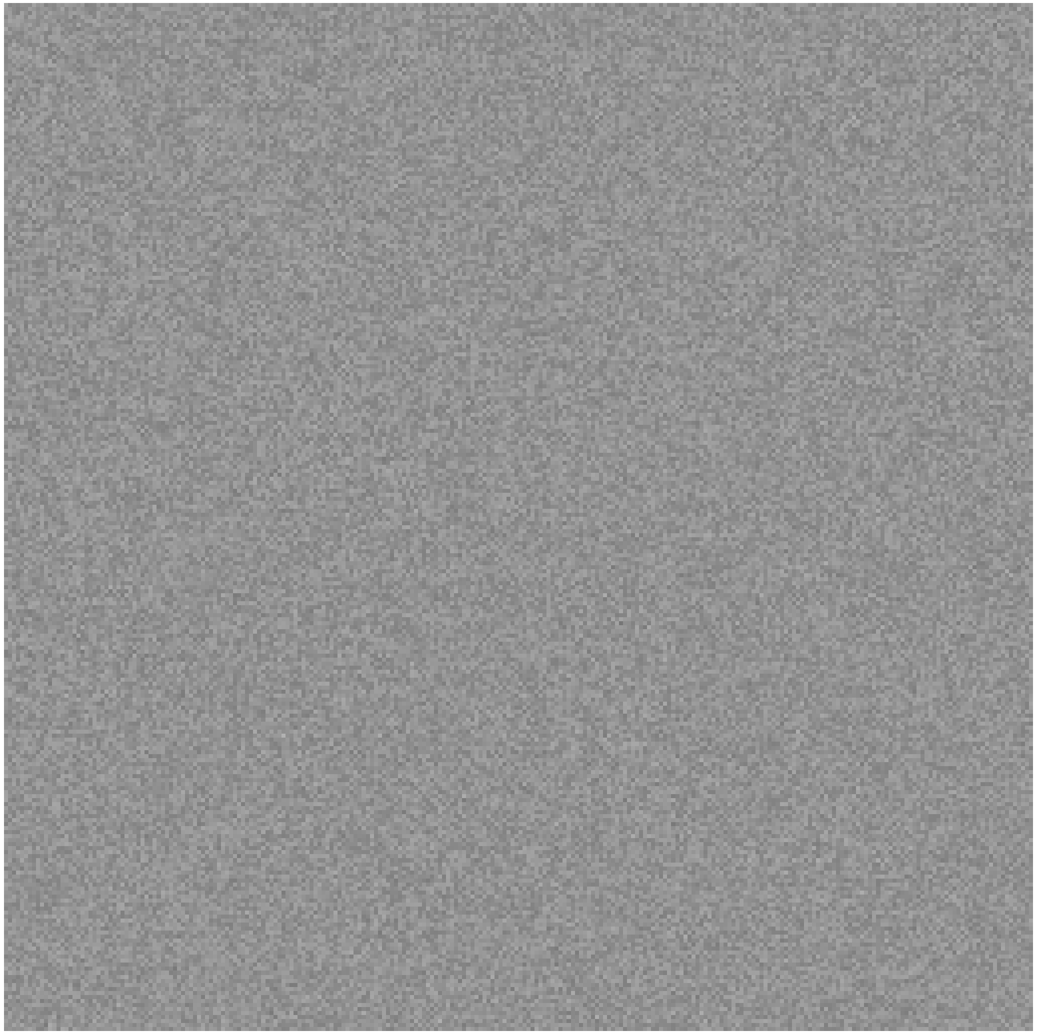
$a = 0$ $D_u = 1, D_v = 5$



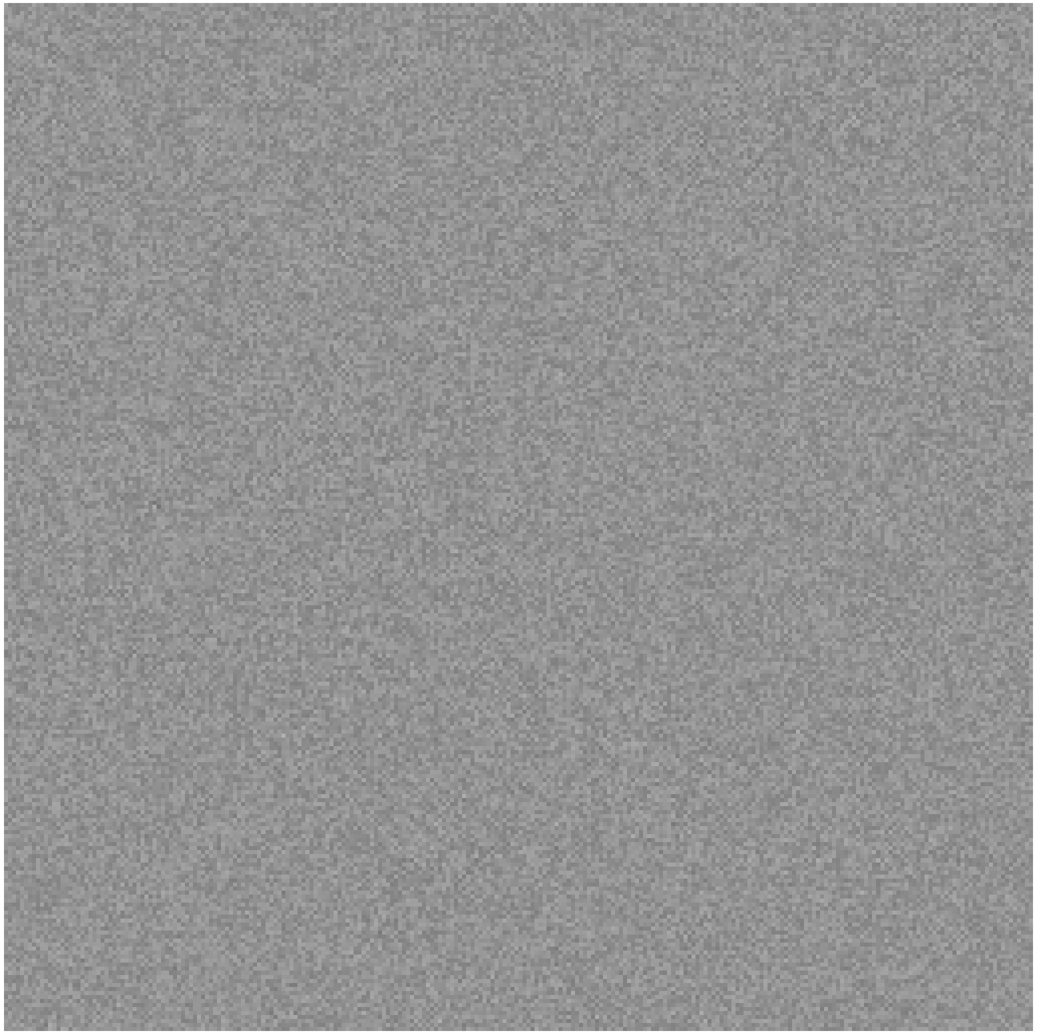
$a = 0$ $D_u = 1, D_v = 20$



$a = 0$ $D_u = 2, D_v = 20$



$a = 0.05$ $D_u = 1, D_v = 20$



Belousov-Zhabotinsky反応

a) Target Pattern

（4倍速）

3 mm

b) Spiral Pattern

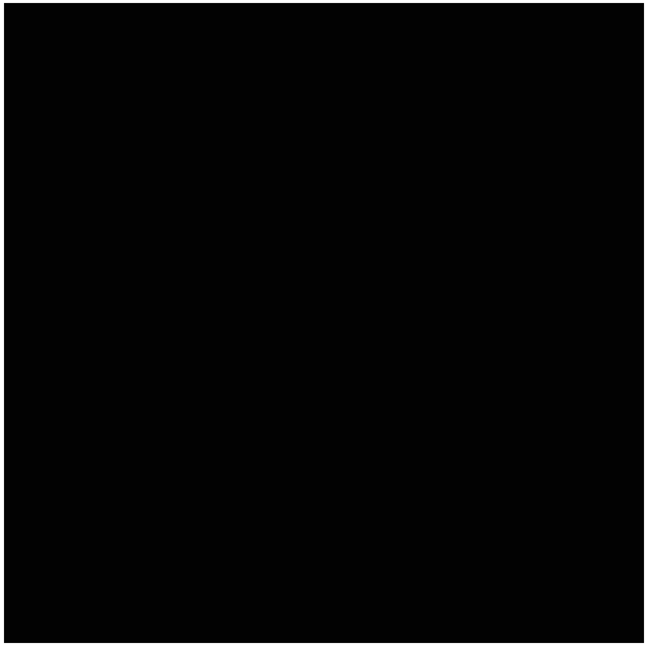
（4倍速）

3 mm

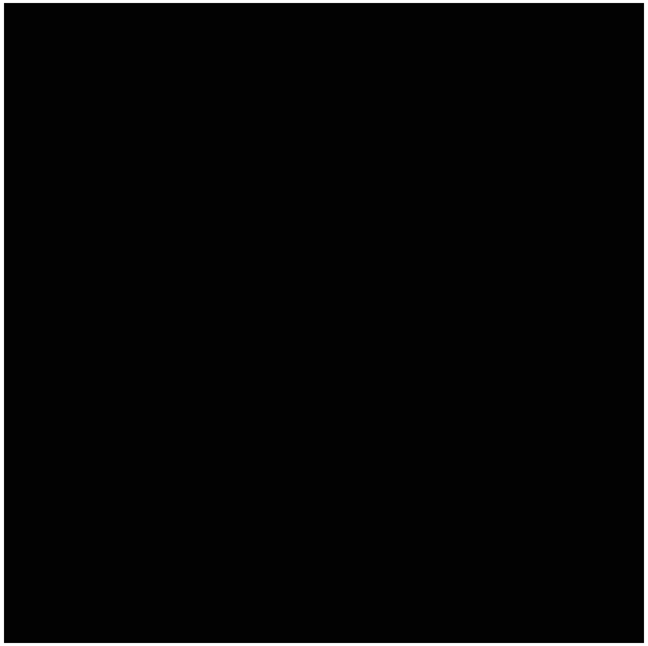
$$\frac{\partial U}{\partial t} = \frac{1}{\varepsilon}\left(U(1-U) - fV\frac{U-q}{U+q} \right) + D_U \nabla^2 U$$

$$\frac{\partial V}{\partial t} = U - V + D_V \nabla^2 V$$

Target Pattern



Spiral Pattern



（Keener-Tyson version Oregonatorを使用）